Euclidean Ramsey Theory

Exposition by William Gasarch (gasarch@cs.umd.edu)

1 Introduction

2 The Square is Ramsey

3 The set of Three Points is Not Ramsey

Let $p = 0$, $q = 1$, $r = 2$ on the real number line. Let $S = \{p, q, r\}$. We show that S is not Ramsey.

We want to show that, for all n, there is a 16-coloring of R^n such that, if $T \subseteq R^n$ is a three points set which is a copy of S then T is not monochromatic. We need to clarify *copy*. Fix n. Let $\vec{p} = (0, \ldots, 0, 0)$, $\vec{q} = (0, \ldots, 0, 1)$, and $\vec{r} = (0, \ldots, 0, 2)$ where there number of coordinates is n.

Definition 3.1 A *copy of S* is a set of the form $\{\vec{p} + \vec{z}, \vec{q} + \vec{z}, \vec{r} + \vec{z}\}$.

Hence we need a property of a copy that is independent of $\vec{z}$.

Theorem 3.2 *There exists a, b, d such that, for all $\vec{z}$*

$$a((\vec{r}+\vec{z})\cdot(\vec{r}+\vec{z}) - (\vec{q}+\vec{z})\cdot(\vec{q}+\vec{z})) + b((\vec{r}+\vec{z})\cdot(\vec{r}+\vec{z}) - (\vec{p}+\vec{z})\cdot(\vec{p}+\vec{z})) = d.$$

Proof:

We derive conditions for a, b, d and then give values that satisfy those conditions. We want

$$a\big((\vec{r}+\vec{z})\cdot(\vec{r}+\vec{z}) - (\vec{q}+\vec{z})\cdot(\vec{q}+\vec{z})) + b((\vec{r}+\vec{z})\cdot(\vec{r}+\vec{z}) - (\vec{p}+\vec{z})\cdot(\vec{p}+\vec{z})) = d.$$

Let $z = (z_1, \ldots, z_n)$. Then the above becomes

$$a\big((\sum_{i=1}^{n-1} z_i^2 + (z_n+2)^2) - (\sum_{i=1}^{n-1} z_i^2 + (z_n+1)^2\big) + b\big(\sum_{i=1}^{n-1} z_i^2 + (z_n+2)^2) - (\sum_{i=1}^{n-1} z_i^2 + z_n^2\big) = d$$

$$a\big((z_n+2)^2 - (z_n+1)^2\big) + b\big((z_n+2)^2 - z_n^2\big) = d$$

$$a(z_n^2 + 4z_n + 4 - z_n^2 - 2z_n - 1) + b(z_n^2 + 4z_n + 4 - z_n^2) = d$$

$$a(2z_n + 3) + b(4z_n + 4) = d$$

$$3a + 4b + (2a + 4b)z_n = d$$

We need to make $2a + 4b = 0$. We take $a = 2$ and $b = -1$. This forces $d = 2$. ∎

We can now rephrase the question (we pre-apologize for using $\vec{z}$ over again in a different context, but we are running out of letters). We want to 16-color R^n so that there are no monochromatic $\vec{x}, \vec{y}, vz$ with

$$a(\vec{x}\cdot\vec{x} - \vec{y}\cdot\vec{y}) + b(\vec{x}\cdot\vec{x} - \vec{z}\cdot\vec{z}) = d.$$

Note that dot products give us reals. Hence we will first give a coloring of the reals and then use it to give a coloring of R^n.

Lemma 3.3 *For all $m \in \mathsf{N}$, for all $\epsilon > 0$ there exists a $2m$-coloring of R such that, for all y, y',*

$$COL(y) = COL(y') \implies y - y' \in \bigcup_{k\in\mathsf{Z}} (2km\epsilon - \epsilon, 2km\epsilon + \epsilon).$$

Proof: We color the reals by coloring intervals of length ϵ that are closed on the left and open on the right. The following picture describe the coloring.

1	2	3	$\cdots$	$2m$	1	2	$\cdots$
$[0,\epsilon)$	$[\epsilon, 2\epsilon)$	$[2\epsilon, 3\epsilon)$	$\cdots$	$[(2m-1)\epsilon, 2m\epsilon)$	$[2m\epsilon, (2m+1)$	$[(2m+1)\epsilon, (2m+2)\epsilon)$	$\cdots$

Assume $COL(y) = COL(y')$. Since we are interested in $y - y'$ we can assume that $y' \in [0,\epsilon)$. If $y > y'$ then

$$y \in [0,\epsilon) \text{ or } y' \in [2m\epsilon, (2m+1)\epsilon) \text{ or } y' \in [4m\epsilon, (4m+1)\epsilon) \text{ or } \cdots.$$

More succintly

$$y \in \bigcup_{k=0}^{\infty} [2km\epsilon, (2km+1)\epsilon)$$

Hence

$$y - y' \in \bigcup_{k=0}^{\infty} ((2km-1)\epsilon, (2km+1)\epsilon)$$

If $y < y'$ then we get, by similar reasioning,

$$y - y' \in \bigcup_{k=0}^{-\infty} ((2km-1)\epsilon, (2km+1)\epsilon)$$

Hence we have

$$y - y' \in \bigcup_{k\in\mathsf{Z}} ((2km-1)\epsilon, (2km+1)\epsilon)$$

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Lemma 3.4 *For all* m*, for all* $a_1, \ldots, z_m \in \mathsf{Z}$*, for all* $d \neq 0$*, there is a* $(2m)^m$ *coloring of* R *such that there there is NO solution to*

$$\sum_{i=1}^{m} a_i(y_i - y_i') = d$$

with $(\forall i)[COL(y_i) = COL(y_i')]$.

Proof: For all $1 \leq i \leq m$ let $\epsilon_i = \frac{d}{a_i m}$ By Lemma 3.3 there exists, for $1 \leq i \leq m$, a coloring COL_i such that

$COL(y) = COL(y')$ implies

$$y - y' \in \bigcup_{k \in \mathsf{Z}} \left(\frac{(2km-1)d}{a_i m}, \frac{(2km+1)d}{a_i m} \right) = \bigcup_{k \in \mathsf{Z}} \left(\frac{2kd}{a_i} - \frac{d}{a_i m}, \frac{2kd}{a_i} + \frac{d}{a_i m} \right)$$

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