Upper Bounds on the Large Ramsey Number $\mathrm{LR}_2(k)$
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1 Introduction

In this manuscript we are only concerned with colorings of the *edges* of a complete graph (and in the appendix the edges of a complete hypergraph). Hence the term *coloring* G will mean coloring the *edges* of G.

Recall Ramsey's theorem:

Theorem 1.1 *For all k, for all c, there exists n such that for any c-coloring of the edges of K_n there exists a set of k points that form a monochromatic clique (that is, all of the edges between the vertices have the same color).*

We need some terminology.

Definition 1.2 Let $n \in \mathsf{N}$. Assume that a complete graph (on a finite or infinite number of vertices) is colored. A *homogeneous set* is a set of vertices of the graph such that every edge between them has the same color. A set is *homogeneous RED* if it is homogeneous and the color is RED (similar for BLUE). A set of size 1 is considered to be homogeneous RED and homogeneous BLUE.

The conclusion of Ramsey's Theorem is the existence of a homogeneous set of size k. What if instead of wanting a homogeneous set of size k we want a homogeneous set that is large compared to the minimum element in the set?

Definition 1.3 Let $A \subseteq \mathsf{N}$. A is *large* if A is at least as big as the minimum element of A.

Example 1.4

1. The set $\{10, 15, 20, \ldots, 100\}$ is large since it has 19 elements and its min element is 10
2. The set $\{10^{10}, 10^{10}+1, \ldots, 10^{10}+10^9\}$ is not large since it has 10^9+1 elements but its min element is 10^{10}.

Consider the following theorem:

Theorem 1.5 *For all c, there exists n such that for any c-coloring of the edges of K_n there exists a large homogeneous set.*

This is a stupid theorem. The set $\{1\}$ is always a large homogeneous set. Even if you don't allow homogeneous sets of size 1 you can take $\{2, 3\}$ to be a large homogeneous set.

How to make this interesting? We need to label the graphs with the vertices $\{k, k+1, \ldots, n\}$. We need some notation and can then state the theorem.

Notation 1.6

1. Let $k, n \in \mathsf{N}$, $k < n$. $K_{[k,n]}$ is the complete graph with vertex set $\{k, k+1, \ldots, n\}$ (This is *not* the complete bipartite graph with k vertices on the left and n vertices on the left even though the notation looks similar.)

2. Let K_ω be the complete graph with vertex set N.

3. Let $K_{[k,\omega)}$ be the complete graph on the vertices $\{k, k+1, \ldots\}$.

The following theorem, called *The Large Ramsey Theorem for Graphs* will be our main concern.

Theorem 1.7 *Let $c \in \mathsf{N}$. For all k there exists n such that for every c-coloring of $K_{[k,n]}$ there exists a large homogeneous set. The number n is called a Large Ramsey Number and is denoted* $\mathrm{LR}_c(k)$.

Paris and Harrington [**?**] proved *The Large Ramsey Theorem for Hypergraphs* which we state in the Appendix. Their interest in it was that it cannot be proven in Peano Arithmetic. The proof that it cannot be proven in Peano Arithmetic hinges on the fact that the Large Ramsey Numbers (for hypergraphs) grow very fast. We give upper bounds in the case of 2-coloring the edges of a graph that are due to Mills [**?**]. Our proof will clearly be in Peano Arithmetic.

In this manuscript we do the following

1. In Section 2 we prove the Large Ramsey Theorem for Graphs. This proof uses infinitary techniques and is not in Peano Arithmetic. It does not provide bounds on $\mathrm{LR}_c(k)$.

2. In Section 3 we define Large Hybrid Ramsey Numbers $\mathrm{LH}(k;m)$ and relate them to the Large Ramsey Numbers.

3. In Section 4 we obtain upper bounds on $\mathrm{LH}(k;m)$.

4. In Section 5 we use the upper bounds on $\mathrm{LH}(k;m)$ to obtain upper bounds on $\mathrm{LR}_2(k)$.

We will need the ordinary Ramsey Numbers. We define them here for completeness

Definition 1.8 $R(a,b)$ is the least number n such that for all 2-colorings of K_n there is either a RED homogeneous set of size a or a BLUE homogeneous set of size b. $R(a,b)$ exists by the ordinary Ramsey's theorem.

2 The Large Ramsey Theorem

We will need the infinite Ramsey Theorem, stated below, to prove the Large Ramsey Theorem.

Theorem 2.1 *For every $c \in \mathsf{N}$, for all c-colorings of K_ω, there exists an infinite homogeneous set.*

Theorem 2.2 *Let $c \in \mathsf{N}$. For all k there exists n such that for every c-coloring of $K_{[k,n]}$ there exists a large homogeneous set.*

Proof:

Assume, by way of contradiction, that the theorem is false. Let c, k be such that for all $n > k$ there exists a c-coloring of $K_{[k,n]}$, which we denote COL_n, such that there is no large homogeneous set relative to COL_n. We use the COL_n's to create a coloring of $K_{[k,\omega)}$ which we call COL.

List out the edges of $K_{[k,\omega)}$: $e_1, e_2, \ldots$. We color each edge as follows.

Initially set

$$\begin{aligned} A_1 &= \mathsf{N} \\ \mathrm{COL}(e_1) &= \text{the least } d \text{ such that } [\exists^\infty n \in A_1 \text{ such that } \mathrm{COL}_n(e_1) = d] \end{aligned}$$

Let $i \geq 2$. Assume inductively that

1. $e_1, \ldots, e_{i-1}$ are colored.
2. $A_i = \{j : \mathrm{COL}_j(e_1) = \mathrm{COL}(e_1), \ldots, \mathrm{COL}_j(e_{i-1}) = \mathrm{COL}(e_{i-1})\}$.
3. A_i is infinite.

Let

$$\begin{aligned} \mathrm{COL}(e_i) &= \text{the least } d \text{ such that } [\exists^\infty n \in A_i \text{ such that } \mathrm{COL}_n(e_i) = d] \\ A_{i+1} &= A_i \cap \{n : \mathrm{COL}_n(e_i) = d\}. \end{aligned}$$

It is easy to see that, after each step, the conditions $1, 2, 3$ still hold.

COL is a c-coloring of $K_{[k,\omega)}$. By Theorem 2.1 there is an infinite homogeneous set

$$H = \{v_1 < v_2 < v_3 < \cdots\}.$$

Take the first v_1 vertices,

$$H' = \{v_1 < v_2 < v_3 < \cdots < v_{v_1}\}.$$

This is a homogeneous set relative to COL. By the definition of COL there is at least one (in fact infinitely many) n such that COL_n agrees with COL on all of the edges between elements of H'. Hence H' is a large homogeneous subset relative to COL_n. This contradicts the definition of COL_n. ∎

Definition 2.3 Let $\mathrm{LR}_c(k)$ be the least n such that, for any c-coloring of $K_{[k,n]}$, there is a large homogeneous set. Note that, for all k, c, $\mathrm{LR}_c(k)$ exists by Theorem 2.2.

3 A Useful Large Hybrid Ramsey Number

Definition 3.1 Let $k \in \mathsf{N}$ and $m \geq 2$. Let $\mathrm{LH}(k; m)$ be the least n such that, for all 2-colorings of $K_{[k,n]}$ there is either a RED K_m or a large BLUE homogeneous set.

Note 3.2

1. LH$(k;1) = k$ given that we say a set of size 1 is homogeneous RED.

2. LH$(k;2) = 2k-1$. If you color $K_{[k,2k-1]}$ and use any RED then you have a RED K_2. If you use no RED then you have a BLUE homogeneous set of size k that includes the element k, hence it is a large BLUE homogeneous set. We use the weaker result LH$(k;2) \le 2k$ in calculations.

Notation 3.3 A *proper* $(m,*)$ *coloring of* $K_{[k,n]}$ is a 2-coloring of $K_{[k,n]}$ with no RED K_m and no large BLUE homogeneous sets.

We relate LH$(k;m)$ to LR$_2(k)$.
ROHAN- RELOOK AT THE THEOREM BELOW.

Theorem 3.4 LR$_2(k) \le$ LH$(k+1;2k-3)$

Proof:
Let $n =$ LH$(k+1;2k-3)$. Assume, by way of contradiction, that there is a proper coloring of $K_{[k,n]}$. (By proper we mean there are no large homogenous sets.) Denote it COL.

KEY- you probably think I am going to restrict the coloring to $K_{[k+1,n]}$ and use that $n =$ LH$(k+1;2k-1)$. I am NOT going to do that! I will instead construct a *different* coloring COL' of $K_{[k+1,n]}$.

Let $x, y \in \{k+1,\ldots,n\}$. We define $COL'(x,y)$ as follows.

1. If $COL(k,x) = COL(k,y) = RED$ then $COL'(x,y) = COL(x,y)$.

2. If $COL(k,x) = COL(k,y) = BLUE$ then $COL'(x,y) = OPP(COL(x,y))$. (The opposite— if $COL(x,y)$ is RED then $COL'(x,y)$ is BLUE and vice versa.)

3. If $COL(k,x) \ne COL(k,y)$ then $COL'(x,y) = RED$. (Gee— we just ignore COL altogether. That seems weird but it actually works.)

By the definition of n there is (with respect to COL') either a RED K_{2k-3} or a large BLUE homogeneous set.

Case 1: There is a large BLUE homogeneous set H (wrt COL') and, for every $x \in H$, $COL(k,x) = RED$ (this is COL, the original coloring). By the definition of COL' H is also a large BLUE homogeneous set wrt COL.

Case 2: There is a large BLUE homogeneous set H (wrt COL') and, for every $x \in H$, $COL(k,x) = BLUE$ (this is COL, the original coloring). By the definition of COL' H is a large RED homogenous set wrt COL. This is very cute- COL' and COL differ on *every* edge in H, but that just makes the set that was large BLUE homogeneous wrt COL' now large RED homogeneous wrt COL.

Case 3: There is a large BLUE homogeneous set H and there exists $x, y \in H$ such that $COL(k,x) \ne COL(k,y)$ (this is COL, the original coloring). AH-HA- this cannot happen since then $COL'(x,y) = RED$.

Case 4: There is a RED K_{2k-3}. Look at $COL(k,x)$ for all x in the RED K_{2k-3}. There are either $k-1$ x's such that $COL(k,x) = RED$ or there are $k-1$ x's such that $COL(k,x) = BLUE$.

Subcase 4a: There are at least $k-1$ x's such that $COL(k,x) = RED$. Let the set of those x's be H'. By the definition of COL' we have that for all $x, y \in H'$, $COL(x,y) = COL'(x,y) = RED$.

Let $H = H' \cup \{k\}$. H is a is a RED homogeneous set of size k that contains k. Hence H is a large RED homogeneous set wrt COL.
Subcase 4b: There are at least k x's such that $COL(k, x) = BLUE$. By the definition of COL' we have that for all $x, y \in H'$, $COL(x, y) \neq COL'(x, y)$ hence, for all $x, y \in H'$, $COL(x, y) = BLUE$. Let $H = H' \cup \{k\}$. H is a is BLUE homogeneous set of size k that contains k. Hence H is a large BLUE homogeneous set wrt COL. ∎

We can also lower bound $\mathrm{LR}_k(2)$ with a Large Hybrid Ramsey Number.

Theorem 3.5 $\mathrm{LR}_2(k) \geq \mathrm{LH}(k; k)$.

Proof: Let $n = \mathrm{LR}_2(k)$. Let COL be a 2-coloring of $K_{[k,n]}$. We show that there is either a RED K_k or a large BLUE homogeneous set. By the definition of n one of the two occurs.

1. There is a large RED homogeneous set. Since the least vertex is labeled k, this is also RED homogeneous K_k.
2. There is a large BLUE homogeneous set. Then there is a large BLUE homogeneous set (duh).

∎

4 Upper Bounds on $\mathrm{LH}(k; m)$

Theorem 4.1 *For $m, n \geq 2$, $\mathrm{LH}(k; m + n) \leq \mathrm{LH}((\mathrm{LH}(k; m))^{m+n-1}; n)$.*

Proof: Let

$$\begin{aligned} w &= \mathrm{LH}(k; m) \\ u &= w^{m+n-1} \\ z &= \mathrm{LH}(u; n) \end{aligned}$$

Picture them placed

$k - w - - - - - - - - - u - - - - - - - - - - - - - - - - z$

Assume, by way of contradiction, that COL is a proper $(m + n, *)$ coloring of $K_{[k,z]}$. Note that there are no RED K_{m+n}'s and no large BLUE homogeneous sets.

Restrict the coloring to $K_{[k,w]}$. By the definition of w either (1) there is a large homogeneous BLUE set, in which case we are done, or (2) there is a RED K_m. Henceforth we assume (2).

Let the vertices of the RED K_m be $p_1 < \cdots < p_m$. Picture

$$k \leq p_1 < p_2 < \cdots < p_m \leq w.$$

$k - - p_1 - - - p_2 - - - p_3 - - - - p_4 - - ... - - p_m - - - - w - - - - - - - - - u - - - - - - - - - - - - - - - - z$

All of the edges between the p's are RED. Look at all of the other elements in $[k, z]$ (not just $[k, w]$ but $[k, z]$). We classify them in terms of the colors going from the p's to them.

Let

$$A_0 = \{x \in [k, z] : COL(p_1, x) = BLUE\}.$$

Note that by the definition of COL A_0 cannot have a RED K_{m+n} (This is independent of the fact that $COL(p_1, x) = BLUE$. There just can't be *any* RED K_{m+n} anywhere.) Note also that A_0 cannot have a BLUE K_{p_1-1} since, that K_{p_1-1} combined with p_1 would yield a large BLUE homogeneous set. Hence

$$|A_0| \leq R(m + n, p_1 - 1).$$

We will not need to be this precise. We will use

$$|A_0| \leq R(m + n, w).$$

Let

$$A_1 = \{x \in [k, z] : COL(p_1, x) = RED \wedge COL(p_2, x) = BLUE\}.$$

Note that A_1 cannot have a RED K_{m+n-1} since this could be combined with p_1 to yield a RED K_{m+n}. Note also that A_1 cannot have a BLUE K_{p_2-1} since, that K_{p_2-1} combined with p_2 would yield a large BLUE homogeneous set. Hence

$$|A_1| \leq R(m + n - 1, p_2 - 1).$$

We will not need to be this precise. We will use

$$|A_1| \leq R(m + n - 1, w).$$

For $1 \leq i \leq m - 1$ Let

$$A_i = \{x \in [k, z] : COL(p_1, x) = \cdots = COL(p_{i-1}, x) = RED \wedge COL(p_i, x) = BLUE\}.$$

Note that A_i cannot have a RED K_{m+n-i} since this could be combined with $p_1, \ldots, p_i$ to yield a RED K_{m+n}. Note also that A_i cannot have a BLUE K_{p_i-1} since, that K_{p_i-1} combined with p_i would yield a large BLUE homogeneous set. Hence

$$|A_i| \leq R(m + n - i, p_i - 1).$$

We will not need to be this precise. We will use

$$|A_i| \leq R(m + n - i, w).$$

Is $[k, z] = \{p_1, \ldots, p_m\} \cup A_0 \cup \cdots \cup A_{m-1}$? NO! We need to define one more set A_m Unfortunately we cannot bound $|A_m|$.

Let

$$A_m = \{x \in [k, z] : COL(p_1, x) = \cdots = COL(p_m, x) = RED\}.$$

I know what you are thinking. You are thinking *we need an upper bound on* $|A_m|$. There might be a way to finish the proof that way; however, we do not know one. We are going to get a *Lower Bound* on $|A_m|$ and then use it in a clever way.

Note that we do have

$$[k, z] = \{p_1, \ldots, p_m\} \cup A_0 \cup \cdots \cup A_m$$

We REALLY want to use those Ramsey bounds on the $|A_i|$ with $1 \le i \le m-1$. We can do that by taking A_m away from the LHS.

$$[k, z] - A_m = \{p_1, \ldots, p_m\} \cup A_0 \cup \cdots \cup A_{m-1}$$

$$|[k, z] - A_m| = |\{p_1, \ldots, p_m\}| + |A_0| + \cdots + |A_{m-1}|$$

By our bounds on $|A_i|$ we have

$$|[k, z] - A_m| \le m + \sum_{i=1}^{m-1} R(m+n-i, w).$$

By Fact 9.3 (in Appendix)

$$(\forall a, b \ge 3)[R(a, b) \le b^{a-1} - b^{a-2}].$$

We want to apply this to $R(m+n-i, w)$ for $1 \le i \le m-1$. We have that $w \ge k \ge 3$. We need $m+n-(m-1) \ge 3$, so we need $n \ge 2$ which we have.

We obtain

$$|[k, z] - A_m| \le m + \sum_{i=1}^{m-1} w^{m+n-i-1} - w^{m+n-i-2}.$$

WOW- LOTS of stuff cancels! So we get

$$|[k, z] - A_m| \le m + w^{m+n-1} - w^{n-1}.$$

By Definition $u = w^{m+n-1}$. Hence we have

$$|[k, z] - A_m| \le m + u - w^{n-1}.$$

One would think that w^{n-1} would be MUCH larger than m so we could bound by just u. While this is true we need something slightly more refined.

Recall that

$$w = \mathrm{LH}(k; m) \ge k + m - 2.$$

(The $\ge$ follows by coloring $K_{[k,k+m-2]}$ with every single edge RED.)

$$w^{n-1} \ge (k + m - 2)^{n-1} \ge k + m.$$

(The last $\ge$ is a ridiculously weak inequality but its all we need.)

So we get

$$|[k,z]-A_m| \le m+u-w^{n-1} \le m+u-k-m = u-k = |[k,z]-[u,z]|.$$

So

$$|A_m| \ge |[u,z]|.$$

Recall this picture.

$k - - - - - w - u - - - - - - - - - - - - - - - - z$

Since $|A_m| \ge [u,z]$ we can form a map f from $[u,z]$ into A_m that is 1-1 and has $f(x) \le x$. We will now define a coloring COL' on $K_{[u,z]}$ and use the definition of $z = \text{LH}(u;n)$.

For all $x,y \in [u,z]$ let

$$COL'(x,y) = COL(f(x), f(y)).$$

By the definition of z one of the following must happen.

1. There is a RED K_n wrt to COL'. Let H' be the set of vertices. Let

$$H = f(H') \cup \{p_1, \ldots, p_m\}.$$

By the definition of f, COL', and A_m, H is a RED K_{n+m}.

2. There is a large BLUE homogeneous set wrt COL'. Let H' be the set of vertices. Let $H = f(H')$. $|H| = |H'|$, but the least element of H is $\le$ the least element of H'. Hence, since H' is a large homogeneous BLUE set, H is a large homogeneous BLUE set.

■

Let $f(k,m)$ be a function we will define later to bound $\text{LH}(k;m)$. What properties does $f(k,m)$ need?

$$\text{LH}(k;1) = k.$$

So we need

$$k \le f(k,1).$$

$$\text{LH}(k;2) = 2k-1.$$

So we need

$$2k-1 \le f(k,2).$$

Actually we will take

$$2k \leq f(k,2)$$

since that is easier to work with.

We do not have a bound on LH$(k;3)$ (yet). Does Theorem 4.1 give us a bound on LH$(k;3)$? NO! Theorem 4.1 only applies if $m, n \geq 2$ so it gives a bound on LH$(k;4)$:

$$\mathrm{LH}(k;4) = \mathrm{LH}(k;2+2) \leq \mathrm{LH}((\mathrm{LH}(k;2))^3;2) \leq \mathrm{LH}((2k)^3;2) \leq 2(2k)^3 = 16k^3.$$

but not on LH$(k;3)$.

In Section 7, Theorem 7.3, we will show the following.

Theorem:

1. For all k, LH$(k;3) \leq R(k-1,3) + 5k - 7$.
2. For all $k \geq 7$, LH$(k;3) \leq k^2$.

End of Statement of Theorem

We use this bound now.

We will need

$$\mathrm{LH}(k;3) \leq k^2.$$

If you try to do a proof by induction that LH$(k;m) \leq f(k,m)$ by using Theorem 4.1 you will see that you need

$$(\forall m \geq 2)[f(f(k,m)^{2m-1}, m) \leq f(k,2m)].$$

and

$$(\forall m \geq 2)[f(f(k,m+1)^{2m}, m) \leq f(k,2m+1)].$$

We could try to *define* an f that satisfies these equations and inequalities. Lets try $f(k,m) = k^{g(m)}$ where we determine g later.

- Since LH$(k;1) = k$, $k^1 \leq f(k,1) = k^{g(1)}$, so we need $g(1) = 1$.
- Since LH$(k;2) = 2k-1$, $k^2 \leq f(k,2) = k^{g(2)}$, so we need $g(2) = 2$. We could have used $k^{\log_k(2k-1)} \leq f(k,2)$ and hence taken $g(2) = 1 + \epsilon(k)$ where $\epsilon(k)$ goes to 0. However, this would be cumbersome and would not improve our results at all.
- Since LH$(k;3) \leq f(k,3) = k^{g(3)}$, so we need $g(3) = 2$.

The recurrences for f above become the following recurrences for g.

$$(\forall m \geq 2)[(k^{g(m)(2m-1)})^{g(m)} \leq k^{g(2m)}]$$

$$(\forall m \geq 2)[k^{g(m)^2(2m-1)} \leq k^{g(2m)}]$$

$$(\forall m \ge 2)[g(m)^2(2m-1) \le g(2m)].$$

and

$$(\forall m \ge 2)[(k^{g(m+1)})^{2m})^{g(m)} \le k^{g(2m+1)}]$$

$$(\forall m \ge 2)[k^{g(m)g(m+1)2m} \le k^{g(2m+1)}]$$

$$(\forall m \ge 2)[g(m)g(m+1)2m \le g(2m+1)].$$

Rather than try to *find* a function g that satisfies this, we *define* a function that satisfies this.

Definition 4.2 Let $g(m)$ be defined as follows.

1. $g(1) = 1$,
2. $g(2) = 2$.
3. $g(3) = 2$.
4. $(\forall m \ge 2)[g(2m) = (2m-1)g(m)^2]$.
5. $(\forall m \ge 2)[g(2m+1) = 2mg(m)g(m+1)]$.

Note 4.3 $g(4) = 3g(2)^2 = 3 \times 4 = 12$

Corollary 4.4 *For all k, for all m,* $\mathrm{LH}(k;m) \le k^{g(m)}$.

Proof: This proof will be easy since g was defined to make it work. We prove this by induction.
Base Case: $m = 1$**.** Need $\mathrm{LH}(k;1) \le k^{g(1)}$. Note that $\mathrm{LH}(k;1) = k$ and $k^{g(1)} = k^1$, so we do have $\mathrm{LH}(k;1) \le k^{g(1)}$.

Induction Step. Assume true for all k and for m. We prove for $2m$ and $2m+1$.
The Corollary for $2m$:
By Theorem 4.1

$$\mathrm{LH}(k;2m) \le \mathrm{LH}((\mathrm{LH}(k;m))^{2m-1};m).$$

By the induction hypothesis

$$\mathrm{LH}(k;m) \le k^{g(m)}$$

hence

$$(\mathrm{LH}(k;m))^{2m-1} \le k^{(2m-1)g(m)}.$$

Therefore

$$\mathrm{LH}((\mathrm{LH}(k;m))^{2m-1};m) \le \mathrm{LH}(k^{(2m-1)g(m)};m).$$

We apply the induction hypothesis again to obtain

$$\mathrm{LH}(k^{(2m-1)g(m)};m) \le (k^{(2m-1)g(m)})^{g(m)} = k^{(2m-1)g(m)^2}$$

By the very definition of g

$$k^{(2m-1)g(m)^2} = k^{g(2m)}.$$

Hence we have

$$\mathrm{LH}(k;2m) \le k^{g(2m)}.$$

The Corollary for $2m+1$:
By Theorem 4.1

$$\mathrm{LH}(k;2m+1) \le \mathrm{LH}((\mathrm{LH}(k;m+1))^{2m};m).$$

By the induction hypothesis

$$\mathrm{LH}(k;m+1) \le k^{g(m+1)}$$

hence

$$(\mathrm{LH}(k;m+1))^{2m} \le k^{2mg(m+1)}.$$

Therefore

$$\mathrm{LH}((\mathrm{LH}(k;m+1))^{2m};m) \le \mathrm{LH}(k^{2mg(m+1)};m).$$

We apply the induction hypothesis again to obtain

$$\mathrm{LH}(k^{2mg(m+1)};m) \le (k^{2mg(m+1)})^{g(m)} = k^{2mg(m+1)g(m)}$$

By the very definition of g

$$k^{2mg(m+1)g(m)} = k^{g(2m+1)}.$$

▮

5 An Upper Bound on $\mathrm{LR}_2(k)$

We can use Theorem 3.4 and Corollary 4.4 to obtain a bound on $\mathrm{LR}_2(k)$ in terms of $g(m)$. We will later obtain upper bounds on g and hence real upper bounds on $\mathrm{LR}_2(k)$.

Corollary 5.1 *Let $k \in \mathsf{N}$. $\mathrm{LR}_2(k) \le (k+1)^{g(2k-3)}$.*

Proof:
By Theorem 3.4

$$\mathrm{LR}_2(k) \le \mathrm{LH}(k+1; 2k-3).$$

By Corollary 4.4

$$\mathrm{LH}(k+1; 2k-3) \le k^{g(2k-3)}.$$

Putting this all together we get

$$\mathrm{LR}_2(k) \le (k+1)^{g(2k-3)}.$$

▮

Now we need to bound $g(m)$.

Lemma 5.2 *If m is a power of 2, $m \ge 2$, then $g(m) \le 2^{m \lg m}$.*

Proof:
1) We prove this by induction on m. Keep in mind that m is a power of 2.
Base Case: $m = 1$. $g(1) = 1$. $2^{1 \lg 1} = 2^0 = 1$. Hence, for $m = 1$, $g(1) \le 2^{1 \lg 1}$.

Induction Step: Assume the lemma for m, prove it for $2m$.

$$g(2m) = (2m-1)g(m)^2 \le (2m-1)2^{2m \lg m} \le 2m \times 2^{2m \lg m} = 2^{2m \lg m + \lg m + 1}.$$

We need

$$2^{2m \lg m + \lg m + 1} \le 2^{2m \lg(2m)}.$$

So we need

$$2m \lg m + \lg m + 1 \le 2m \lg(2m).$$

$$2m \lg m + \lg m + 1 \le 2m(1 + \lg m).$$

$$2m \lg m + \lg m + 1 \le 2m + 2m \lg m).$$

$$\lg m + 1 \le 2m.$$

This is true, so we are done.

▮

We can now get a real bound on $\mathrm{LR}_2(k)$.

Theorem 5.3

$$\mathrm{LR}_2(k) \le (2k+1)^{2^{4k\lg(4k)}}.$$

Proof: Let

$$m = \text{ the least } n \text{ such that } n \ge k \wedge 2n-3 \text{ is a power of 2.}$$

Note that $m \le 2k$. It is easy to show that $\mathrm{LR}_2(x)$ is a monotone increasing function of x. Hence

$$\mathrm{LR}_2(k) \le \mathrm{LR}_2(m).$$

By Corollary 5.1

$$\mathrm{LR}_2(m) \le (m+1)^{g(2m-3)}.$$

Since $2m-3$ is a power of 2 we can use Lemma 5.2 to obtain

$$g(2m-3) \le 2^{(2m-3)\lg(2m-3)} \le 2^{2m\lg(2m)}.$$

Since $m \le 2k$ we have $2^{2m\lg(2m)} \le 2^{4k\lg(4k)}$.
Putting this all together we get

$$\mathrm{LR}_2(k) \le (2k+1)^{2^{4k\lg(4k)}}.$$

∎

6 More Refined Upper Bounds on $\mathrm{LR}_2(k)$

Can we improve the bound on $\mathrm{LR}_2(k)$ from Theorem 5.3? We can! To do this we will improve our upper bound on g.

Definition 6.1 Let $\beta(m)$ be such that $g(m) = 2^{m\beta(m)}$. Note the following.

- Since $2^0 = g(1) = 2^{1\times\beta(1)}$, $\beta(1) = 0$.
- Since $2^1 = g(2) = 2^{2\times\beta(2)}$, $\beta(2) = 1/2$.
- Since $2^1 = g(3) = 2^{3\times\beta(2)}$, $\beta(3) = 1/3$.

Note 6.2 In Lemma 5.2 we showed that $\beta(m) \le \lg(2m)$.

We want to show that β is bounded by a constant.

Definition 6.3 A number is *cool* if it is of the form 3×2^i where $i \ge 0$.

Lemma 6.4

1. *For all* $m \geq 2$, $\beta(2m) = \frac{\lg(2m-1)}{2m} + \beta(m)$.

2. *For all* m *that are cool*

$$\beta(m) \leq 1.471.$$

Proof:
1)

$$g(m) = 2^{m\beta(m)}$$

$$\lg(g(m)) = m\beta(m)$$

$$\beta(m) = \frac{\lg(g(m))}{m}.$$

We now express $\beta(2m)$ in terms of $\beta(m)$.

$$\beta(2m) = \frac{\lg(g(2m))}{2m} = \frac{\lg((2m-1)g(m)^2)}{2m} = \frac{\lg(2m-1)}{2m} + \frac{\lg(g(m))}{m} = \frac{\lg(2m-1)}{2m} + \beta(m).$$

Hence

$$\beta(2m) = \frac{\lg(2m-1)}{2m} + \beta(m).$$

2) By Part 1, for all $m \geq 2$,

$$\beta(2m) = \frac{\lg(2m-1)}{2m} + \beta(m).$$

Letting $m = 3 \times 2^j$ we obtain that, for all $j \geq 0$,

$$\beta(3 \times 2^{j+1}) = \frac{\lg(3 \times 2^{j+1} - 1)}{3 \times 2^{j+1}} + \beta(3 \times 2^j).$$

Let $\gamma(j) = \beta(3 \times 2^j)$. Note that $\gamma(0) = \beta(3) = 1/3$.
We have

$$\gamma(j+1) = \frac{\lg(3 \times 2^{j+1} - 1)}{3 \times 2^{j+1}} + \gamma(j).$$

Hence, by Lemma 9.4 (in the Appendix), for all j,

$$\gamma(j) \leq \gamma(0) + \sum_{i=1}^{\infty} \frac{\lg(3 \times 2^i - 1)}{3 \times 2^i}.$$

Note that $\gamma(0) = 1/3$ is of the form for the summation at 0. Hence we have

$$\gamma(j) \leq \sum_{i=0}^{\infty} \frac{\lg(3 \times 2^i - 1)}{3 \times 2^i}.$$

We will take out the first B terms (B to be determined later) to obtain

$$\gamma(j) \leq \sum_{i=0}^{B} \frac{\lg(3 \times 2^i - 1)}{3 \times 2^i} + \sum_{i=B+1}^{\infty} \frac{\lg(3 \times 2^i - 1)}{3 \times 2^i}$$

Note that

$$\frac{\lg(3 \times 2^i - 1)}{3 \times 2^i} \leq \frac{\lg 3}{3 \times 2^i} + \frac{i}{3 \times 2^i} \leq \frac{\lg 3}{3}\frac{1}{2^i} + \frac{1}{3}\frac{i}{2^i}.$$

Hence

$$\sum_{i=B}^{\infty} \frac{\lg(3 \times 2^i - 1)}{3 \times 2^i} \leq \frac{\lg 3}{3} \sum_{i=B}^{\infty} \frac{1}{2^i} + \frac{1}{3} \sum_{i=B}^{\infty} \frac{i}{2^i}.$$

It is well known that

$$\sum_{i=B+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^B}.$$

By Lemma 9.5 in the Appendix we have that

$$\sum_{i=B+1}^{\infty} \frac{i}{2^i} = \frac{B+2}{2^B}.$$

Hence

$$\frac{\lg 3}{3} \sum_{i=B}^{\infty} \frac{1}{2^i} + \frac{1}{3} \sum_{i=B}^{\infty} \frac{i}{2^i} \leq \frac{\lg 3 + B + 2}{3 \times 2^B}.$$

Therefore

$$\beta(3 \times 2^j) \leq \gamma(j) \leq \frac{\lg 3 + B + 2}{3 \times 2^B} + \sum_{i=0}^{B} \frac{\lg(3 \times 2^i - 1)}{3 \times 2^i}$$

A short MATLAB program computed this for $B = 1$ to 100. The number seemed to converge to 1.4706, so we bound by 1.471. Hence

$$\beta(3 \times 2^j) \leq 1.471.$$

▮

Note 6.5 Mills [?] claims that, for all m, $\beta(m) \leq 1.471$.

Note 6.6 The reason we took $\gamma(j) = \beta(3 \times 2^j)$ was because we knew $\beta(3) = \gamma(0)$. What if we knew $\beta(L)$? Then we could take $\gamma(j) = \beta(L \times 2^j)$ and we would have $\gamma(0) = \beta(L)$. If we work this through we would end up with the following:

$$\beta(L \times 2^j) \leq \gamma(j) \leq \frac{\lg L + B + 2}{L \times 2^B} + \beta(L) + \sum_{i=1}^{B} \frac{\lg(L \times 2^i - 1)}{L \times 2^i}$$

This might lead to a constant better than 1.471.

Theorem 6.7

$$\mathrm{LR}_2(k) \le (2k+1)^{2^{5.884k}}.$$

Proof: Let

$$m = \text{ the least } n \text{ such that } n \ge k \wedge 2n-3 \text{ is cool.}$$

Note that $m \le 2k$. It is easy to show that $\mathrm{LR}_2(x)$ is a monotone increasing function of x. Hence

$$\mathrm{LR}_2(k) \le \mathrm{LR}_2(m).$$

By Corollary 5.1

$$\mathrm{LR}_2(m) \le (m+1)^{g(2m-3)}.$$

Since $2m-3$ is cool use Lemma 6.4 to obtain

$$g(2m-3) \le 2^{1.471(2m-3)} \le 2^{1.471\times 2m} \le 2^{2.942m}.$$

Since $m \le 2k$ we have $2^{2.942m} \le 2^{5.884k}$. Hence

$$\mathrm{LR}_2(k) \le (2k+1)^{2^{5.884k}}.$$

■

7 Upper Bounds on $\mathrm{LH}(k;3)$

Definition 7.1 Let $k \le n$. Let COL be a 2-coloring of $K_{[k,n]}$. If $v \in \{k,\ldots,n\}$ then

$$RED(v) = \{w : COL(v,w) = RED\}$$

$$BLUE(v) = \{w : COL(v,w) = BLUE\}$$

Lemma 7.2 *Let $k \le n$. Let COL be a proper $(3,*)$-coloring of $K_{[k,n]}$.*

1. $n = k + |RED(k)| + |BLUE(k)|$.
2. *For all* $v \in \{k,\ldots,v\}$, $|BLUE(v)| \le R(3, v-1) - 1$.
3. *For all* $v < w$, $|RED(v) \cap BLUE(w)| \le w - 2$

Proof:
1) There are $n-k$ vertices in the graph. For each vertex v either $v \in RED(k)$ or $v \in BLUE(k)$ but not both. Hence

$$n - k = |RED(k)| + |BLUE(k)$$

$$n = k + |RED(k)| + |BLUE(k)|$$

2) Assume, by way of contradiction, that $|BLUE(v)| \geq R(3, v-1)$. Look at COL restricted to $BLUE(v)$. Since $|BLUE(v)| \geq R(3, v-1)$ either this graph has a RED K_3 (contradicting COL being a proper $(3, *)$-coloring), or this graph has a BLUE K_{v-1}. In the later case let H be the set of vertices in the BLUE K_{v-1}. Since for all $x \in H$, $COL(x, v) = BLUE$, $H \cup \{v\}$ form a BLUE K_v. Note that v is one of the vertices in this BLUE clique, so it is a large BLUE homogeneous set.

3) Let $X = RED(v) \cap BLUE(w)$. Assume, by way of contradiction, that $|X| \geq w - 1$. For all $x, y \in X$ we have $COL(x, y) = BLUE$, else there would be a RED K_3 with v, x, y. Hence the vertices of X form a BLUE K_{w-1}. Add in the vertex w and you have a BLUE K_w with vertex w, so it is a large BLUE homogeneous set. ∎

Theorem 7.3

1. *For all k,* $\mathrm{LH}(k; 3) \leq R(k-1, 3) + 5k - 7$.
2. *For all $k \geq 7$,* $\mathrm{LH}(k; 3) \leq k^2$.

BILL- LOOK INTO WHAT HAPPENS WHEN $k = 1, 2, 3, 4, 5, 6$.

Proof:

1) Let $n = R(k-1, 3) + 5k - 7$. Assume, by way of contradiction, that there is a proper $(3, *)$ coloring of $K_{[k,n]}$. Call it COL. By Lemma 7.2 parts a and b

$$R(k-1, 3) + 5k - 7 = n = k + |RED(k)| + |BLUE(k)| \leq k + |RED(k)| + R(k-1, 3) - 1$$

$$5k - 7 \leq k + |RED(k)| - 1$$

$$4k - 6 \leq |RED(k)|.$$

Let $x = \min\{RED(k)\}$.

There are two cases.

Case 1: $x \leq 4k - 6$. Note that all of the edges between elements of $RED(k)$ must be BLUE (else you get a RED K_3). Hence the elements of $RED(k)$ form a BLUE homogeneous set of size $|RED(k)| \geq 4k - 6$. If $x \leq 4k - 6$ then this BLUE homogeneous set has an element that is $\leq 4k - 6$ and is hence a large homogeneous BLUE set.

Case 2: $x \geq 4k - 5$ (actually all we need is $x \geq 2k + 1$). Note that, for all $x \in \{k+1, \ldots, 2k\}$, $COL(k, x) = BLUE$. Hence there must be a, b with $k \leq a < b \leq 2k-1$ such that $COL(a, b) = RED$ (else the vertices $\{k, \ldots, 2k-1\}$ form a large BLUE homogeneous set).

We now upper bound $|RED(k)|$. Since $COL(a, b) = RED$, for every x either $COL(a, x) = BLUE$ or $COL(b, x) = BLUE$ (or else you have a RED K_3). Hence

$$|RED(k)| \leq |RED(k) \cap BLUE(a)| + |RED(k) \cap BLUE(b)|.$$

By Lemma 7.2.3 and $a \leq 2k - 2$ and $b \leq 2k - 1$ we have

$$|RED(k)| \le (a-2)+(b-2) \le a+b-4 \le (2k-2)+(2k-1)-4 = 4k-7.$$

By Lemma 7.2 and the bound on $|RED(k)|$

$$n \le k + |RED(k) + |BLUE(k)| \le k + 4k - 7 + R(3, k-1) - 1 = 5k - 8 + R(3, k-1).$$

This contradicts the definition of n.

2) By Fact 9.2

$$R(a, b) \le \binom{a+b-2}{b-1}.$$

Hence

$$R(k-1, 3) \le \binom{k}{2} = \frac{k(k-1)}{2}.$$

This can be used to show that, for $k \ge 7$,

$$R(k-1, 3) + 5k - 7 \le k^2.$$

∎

8 Appendix on Large Ramsey For Hypergraphs

Notation 8.1

1. Let $k, n \in \mathsf{N}$, $k < n$. $K^m_{[k,n]}$ is the complete m-hypergraph with vertex set $\{k, k+1, \ldots, n\}$. (This looks like the notation for the cross product of graphs or an m-partite graph, but its not.)

2. Let K^m_ω be the complete m-hypergraph with vertex set N.

3. Let $K^m_{[k,\omega)}$ be the complete m-hypergraph with vertex set $\{k, k+1, \ldots\}$.

4. We will only be coloring EDGES of complete m-hypergraphs. Henceforth in this manuscript the term *coloring* G will mean coloring the *edges* of G.

5. Assume that a complete m-hypergraph (on a finite or infinite number of vertices) is colored. A *homogeneous set* is a set of vertices of the graph such that every edge between them has the same color.

6. Let $A \subseteq \mathsf{N}$. A is *large* if A is larger than its minimal element. (Same as in main paper.)

Recall the infinite hypergraph Ramsey Theorem:

Theorem 8.2 *For every $m \in \mathsf{N}$, for every $c \in \mathsf{N}$, for all c-colorings of K^m_ω, there exists an infinite homogeneous set.*

This can be used to give a proof of the Large Ramsey Theorem:

Theorem 8.3 *For every $m \in \mathsf{N}$, for every $c \in \mathsf{N}$, for all k there exists n such that for every c-coloring of $K^m_{[k,n]}$ there exists a large homogeneous set.*

We omit the proof, though it is similar to the proof of Theorem 2.1.

Definition 8.4 Let $LR^m_c(k)$ be the n in Theorem 8.3.

Paris and Harrington showed that Theorem 8.3 cannot be proven in Peano Arithmetic. They showed that the function $LR^m_c(k)$ grows faster than any function whose existence can be proven in Peano Arithmetic. This essentially means that the proof from the Theorem 8.2 is really the only proof- so to prove this finitary theorem requires infinitary techniques. So a proof like that of the original finite Ramsey Theorem, or of the bound in $\mathrm{LR}_2(k)$ in this manuscript, cannot be obtained for Theorem 8.3.

9 Appendix: Some Misc Facts

The following recurrence is well known. It is the key to one of the proofs of the finite Ramsey Theorem.

Fact 9.1

$$R(a,b) \le R(a,b-1) + R(a-1,b) - 1.$$

Fact 9.2

$$R(a,b) \le \binom{a+b-2}{a-1} = \binom{a+b-2}{b-1}.$$

Proof:

BILL- INSERT.

▮

Fact 9.3 *If $3 \le a \le b$ then*

$$R(a,b) \le b^{a-1} - b^{a-2}.$$

Proof:

We prove this by induction on $a+b$.

Base Case: Since $3 \le a \le b$ the least $a+b$ can be is 6. If $a+b=6$ and $3 \le a \le b$ then $a=3$ and $b=3$. It is known that $R(3,3)=6$. $b^{a-1} - b^{a-2} = 3^2 - 3^1 = 6$.

Induction Step: Assume the statement is true for any $3 \le a' \le b'$ such that $a'+b' < a+b$. By Fact 9.1

$$R(a,b) \le R(a,b-1) + R(a-1,b).$$

There are two cases.

Case 1: $a \le b-1$. Then we can use the induction hypothesis and obtain

$$R(a,b) \le R(a,b-1) + R(a-1,b) \le (b-1)^{a-1} - (b-1)^{a-2} + b^{a-2} - b^{a-3}.$$

Now we need

$$\begin{aligned}
(b-1)^{a-1} - (b-1)^{a-2} + b^{a-2} - b^{a-3} &\le b^{a-1} - b^{a-2} \\
(b-1)^{a-1} - (b-1)^{a-2} &\le b^{a-1} - 2b^{a-2} + b^{a-3} \\
(b-1)^{a-2}(b-1-1) &\le b^{a-3}(b^2 - 2b + 1) \\
(b-1)^{a-2}(b-2) &\le b^{a-3}(b-1)^2 \\
(b-1)^{a-4}(b-2) &\le b^{a-3}
\end{aligned}$$

GREAT- This is TRUE!

Case 2: $a = b$.

$$R(a,a) \le R(a,a-1) + R(a-1,a) = 2R(a-1,a) \le a^{a-1} - a^{a-2}.$$

This is exactly what we need!. ▮

Lemma 9.4 *Let* $\gamma : \mathsf{N} \to \mathsf{R}$*. Let* $\alpha : \mathsf{N} \to \mathsf{R}$*. Assume that, for all* $j \ge 0$*,*

$$\gamma(j+1) \le \gamma(j) + \alpha(j+1).$$

Then, for all $j \ge 0$*,*

$$\gamma(j) \le \gamma(0) + \sum_{i=1}^{\infty} \alpha(i).$$

Proof:
$\gamma(j+1) - \gamma(j) \le \alpha(j+1)$
$\gamma(j) - \gamma(j-1) \le \alpha(j)$
$\vdots$
$\gamma(2) - \gamma(1) \le \alpha(2)$
$\gamma(1) - \gamma(0) \le \alpha(1)$
If you add up these equations you get
$\gamma(j+1) - \gamma(0) \le \sum_{i=1}^{\infty} \alpha(i)$.
Hence, for all j,
$\gamma(j) \le \gamma(0) + \sum_{i=1}^{\infty} \alpha(i)$. ▮

Lemma 9.5

1. *If* $0 < x < 1$ *then*

$$\sum_{i=0}^{\infty} ix^i = \frac{x}{(1-x)^2}$$

2. $$\sum_{i=0}^{\infty} \frac{i}{2^i} = 2$$

(This follows from part a by plugging in $x = 1/2$*.)*

3.

$$\sum_{i=0}^{B} \frac{i}{2^i} = \frac{2^{B+1} - B - 2}{2^B}.$$

4.

$$\sum_{i=B+1}^{\infty} \frac{i}{2^i} = \frac{B+2}{2^B}.$$

(This follows from parts 2 and 3.)

Proof:
1 and 2) For any x with $0 < x < 1$ we know that

$$\sum_{i=0}^{\infty} x^i = \frac{-1}{x-1}$$

Differentiate this to obtain

$$\sum_{i=0}^{\infty} i x^{i-1} = \frac{1}{(1-x)^2}$$

Multiply both sides by x to obtain

$$\sum_{i=0}^{\infty} i x^{i} = \frac{x}{(1-x)^2}$$

Plug in $x = 1/2$ to obtain the result.

3 and 4)
For any x with $0 < x < 1$ we know that

$$\sum_{i=0}^{B} x^i = \frac{x^{B+1} - 1}{x-1}$$

Differentiate this to obtain

$$\sum_{i=0}^{B} i x^{i-1} = \frac{(x-1)(B+1)x^B - (x^{B+1} - 1)}{(x-1)^2}$$

Multiply by x to obtain

$$\sum_{i=0}^{B} i x^{i} = \frac{x}{(x-1)^2}\left[(x-1)(B+1)x^B - (x^{B+1} - 1)\right] = \frac{x}{(x-1)^2}\left[Bx^{B+1} - (B+1)x^B + 1\right]$$

Plug in $x = 1/2$ to obtain

$$2 \times \left[\frac{B}{2^{B+1}} - \frac{B+1}{2^B} + 1\right] = \frac{B}{2^B} - \frac{2B+2}{2^B} + \frac{2^{B+1}}{2^B} = \frac{2^{B+1} - B - 2}{2^B}.$$

∎

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