Roth's Theorem: If $A \subseteq [n]$ is large then it has a 3-AP
Szemeredi's Proof
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1 Roth's Theorem

Notation 1.1 Let $[n] = \{1, \ldots, n\}$. If $k \in \mathsf{N}$ then k-AP means an arithmetic progression of size k.

Consider the following statement:
If $A \subseteq [n]$ and $|A|$ is 'big' then A must have a 3-AP.

This statement, made rigorous, is true. In particular, the following is true and easy:
Let $n \geq 3$. If $A \subseteq [n]$ and $|A| \geq 0.7n$ then A must have a 3-AP.

Can we lower the constant 0.7? We can lower it as far as we like if we allow n to start later: Roth [2, 4, 5] proved the following using analytic means.
$(\forall \lambda > 0)(\exists n_0 \in \mathsf{N})(\forall n \geq n_0)(\forall A \subseteq [n])[|A| \geq \lambda n \Rightarrow A$ has a 3-AP].

The analogous theorem for 4-APs was later proven by Szemeredi [2, 6] by a combinatorial proof. Szemeredi [7] later (with a much harder proof) generalized from 4 to any k.

We prove the $k = 3$ case using the combinatorial techniques of Szemeredi. Our proof is essentially the same as in the book *Ramsey Theory* by Graham, Rothchild, and Spencer [2].

More is known. A summary of what else is known will be presented in the next section.

Definition 1.2 Let $sz(n)$ be the least number such that, for all $A \subseteq [n]$, if $|A| \geq sz(n)$ then A has a 3-AP. Note that if $A \subseteq [a, a+n-1]$ and $|A| \geq sz(n)$ then A has a 3-AP. Note also that if $A \subseteq \{a, 2a, 3a, \ldots, na\}$ and $|A| \geq sz(n)$ then A has a 3-AP. More generally, if A is a subset of any equally spaced set of size n, and $|A| \geq sz(n)$, then A has a 3-AP.

We will need the following Definition and Lemma.

Definition 1.3 Let $k, e, d_1, \ldots, d_k \in \mathsf{N}$. The *cube on* $(e, d_1, \ldots, d_k)$, denoted $C(e, d_1, \ldots, d_k)$, is the set $\{e + b_1 d_1 + \cdots + b_k d_k \mid b_1, \ldots, b_k \in \{0, 1\}\}$. A *k-cube* is a cube with k d's.

Lemma 1.4 *Let I be an interval of $[1, n]$ of length L. If $|B| \subseteq I$ then there is a cube $(e, d_1, \ldots, d_k)$ contained in B with $k = \Omega(\log \log |B|)$ and $(\forall i)[d_i \leq L]$.*

Proof:
The following procedure produces the desired cube.

1. Let $B_1 = B$ and $\beta_1 = |B_1|$.
2. Let D_1 be all $\binom{\beta_1}{2}$ positive differences of elements of B_1. Since $B_1 \subseteq [n]$ all of the differences are in $[n]$. Hence some difference must occur $\binom{\beta_1}{2}/n \sim \beta_1^2/2n$ times. Let that difference be d_1. Note that $d_1 \leq L$.

3. Let $B_2 = \{x \in B_1 : x + d_1 \in B_1\}$. Note that $|B_2| \geq \beta_1^2/2n$. Let $|B_2| = \beta_2$. Note the trivial fact that

   $x \in B_1 \Rightarrow x + d_1 \in B$.

4. Let D_2 be all $\binom{\beta_2}{2}$ positive differences of elements of B_2. Since $B_2 \subseteq [n]$ all of the differences are in $[n]$. Hence some difference must occur $\binom{\beta_1}{2}/n \sim \beta_2^2/2n$ times. Let that difference be d_2. Note that $d_2 \leq L$.

5. Let $B_3 = \{x \in B_2 : x + d_2 \in B_2\}$. Note that $|B_3| \geq \beta_2^2/2n$. Let $|B_3| = \beta_3$. Note that

   $x \in B_3 \Rightarrow x + d_2 \in B$

   $x \in B_3 \Rightarrow x \in B_2 \Rightarrow x + d_1 \in B$

   $x \in B_3 \Rightarrow x + d_2 \in B_2 \Rightarrow x + d_1 + d_2 \in B$

6. Keep repeating this procedure until $B_{k+2} = \emptyset$. (We leave the details of the definition to the reader.) Note that if $i \leq k+1$ then

   $x \in B_i \Rightarrow x + b_1 d_1 + \cdots + b_{i-1} d_{i-1} \in B$ for any $b_1, \ldots, b_{i-1} \in \{0, 1\}$.

7. Let e be any element of B_{k+1}. Note that we have $e + b_1 d_1 + \cdots + b_k d_k \in B$ for any $b_1, \ldots, b_k \in \{0, 1\}$.

We leave it as an exercise to formally show that $C(e, d_1, \ldots, d_k)$ is contained in B and that $k = \Omega(\log\log|B|)$. ▮

We now note that the above gives a good upper bound on the Hilbert Cube Numbers.

Corollary 1.5 *For k, c let $H(k, c)$ be the least H such that for any c-coloring of $\{1, \ldots, H\}$ there is a monochromatic k-cube. Then $H(k, c) \leq c2^{2^{O(k)}}$.*

Proof: Let $H = c2^{2^{Ak}}$ where we define A later. Let COL be a c-coloring of $\{1, \ldots, H\}$. Some color appears $H/c = 2^{2^{Ak}}$ times. Let B be the set of integers with that color, so $|B| = 2^{2^{Ak}}$. By Lemma 1.4 there is a monochromatic cube of size $\Omega(\log_2(\log_2(|B|))) = \Omega(Ak)$. Pick A big enough so that this term is $\geq k$. ▮

The next lemma states that if A is 'big' and 3-free then it is somewhat uniform. There cannot be sparse intervals of A. The intuition is that if A has a sparse interval then the rest of A has to be dense to make up for it, and it might have to be so dense that it has a 3-AP.

Lemma 1.6 *Let $n, n_0 \in \mathsf{N}; \lambda, \lambda_0 \in (0, 1)$. Assume $\lambda < \lambda_0$ and $(\forall m \geq n_0)[sz(m) \leq \lambda_0 m]$. Let $A \subseteq [n]$ be a 3-free set such that $|A| \geq \lambda n$.*

1. *Let a, b be such that $a < b$, $a > n_0$, and $n - b > n_0$. Then $\lambda_0(b - a) - n(\lambda_0 - \lambda) \leq |A \cap [a, b]|$.*

2. *Let a be such that $n - a > n_0$. Then $\lambda_0 a - n(\lambda_0 - \lambda) \leq |A \cap [1, a]|$.*

Proof:

1) Since A is 3-free and $a \geq n_0$ and $n - b \geq n_0$ we have $|A \cap [1, a-1]| < \lambda_0(a-1) < \lambda_0 a$ and $|A \cap [b+1, n]| < \lambda_0(n-b)$. Hence

$$\begin{aligned} \lambda n \leq |A| = & \ |A \cap [1, a-1]| + |A \cap [a, b]| + |A \cap [b+1, n]| \\ \lambda n \leq & \ \lambda_0 a + |A \cap [a, b]| + \lambda_0(n-b) \\ \lambda n - \lambda_0 n + \lambda_0 b - \lambda_0 a \leq & \ |A \cap [a, b]| \\ \lambda_0(b-a) - n(\lambda_0 - \lambda) \leq & \ |A \cap [a, b]|. \end{aligned}$$

2) Since A is 3-free and $n - a > n_0$ we have $|A \cap [a+1, n]| \leq \lambda_0(n-a)$. Hence

$$\begin{aligned} \lambda n \leq |A| = & \ |A \cap [1, a]| + |A \cap [a+1, n]| \\ \lambda n \leq & \ |A \cap [1, a]| + \lambda_0(n-a) \\ \lambda n - \lambda_0 n + \lambda_0 a \leq & \ |A \cap [1, a]| \\ \lambda_0 a - (\lambda_0 - \lambda) n \leq & \ |A \cap [1, a]|. \end{aligned}$$

▮

Lemma 1.7 *Let $n, n_0 \in \mathsf{N}$ and $\lambda, \lambda_0 \in (0,1)$. Assume that $\lambda < \lambda_0$ and that $(\forall m \geq n_0)[sz(m) \leq \lambda_0 m]$. Assume that $\frac{n}{2} \geq n_0$. Let $a, L \in \mathsf{N}$ such that $a \leq \frac{n}{2}$, $L < \frac{n}{2} - a$, and $a \geq n_0$. Let $A \subseteq [n]$ be a 3-free set such that $|A| \geq \lambda n$.*

1. *There is an interval $I \subseteq [a, \frac{n}{2}]$ of length $\leq L$ such that*

$$|A \cap I| \geq \left\lfloor \frac{2L}{n-2a}(\lambda_0(\frac{n}{2} - a) - n(\lambda_0 - \lambda)) \right\rfloor.$$

2. *Let α be such that $0 < \alpha < \frac{1}{2}$. If $a = \alpha n$ and $\sqrt{n} << \frac{n}{2} - \alpha n$ then there is an interval $I \subseteq [a, \frac{n}{2}]$ of length $\leq O(\sqrt{n})$ such that*

$$|A \cap I| \geq \left\lfloor \frac{2\sqrt{n}}{(1-2\alpha)}(\lambda_0(\frac{1}{2} - (\lambda_0 - \lambda) - \alpha)) \right\rfloor = \Omega(\sqrt{n}).$$

Proof: By Lemma 1.6 with $b = \frac{n}{2}$, $|A \cap [a, \frac{n}{2}]| \geq \lambda_0(\frac{n}{2} - a - n(\lambda_0 - \lambda)$. Divide $[a, \frac{n}{2}]$ into $\left\lceil \frac{n-2a}{2L} \right\rceil$ intervals of size $\leq L$. There must exist an interval I such that

$$|A \cap I| \geq \left\lfloor \frac{2L}{n-2a}(\lambda_0(\frac{n}{2} - a) - n(\lambda_0 - \lambda)) \right\rfloor.$$

If $L = \lceil \sqrt{n} \rceil$ and $a = \alpha n$ then

$$\begin{aligned} |A \cap I| \ & \geq \left\lfloor \tfrac{2L}{n-2a}(\lambda_0(\tfrac{n}{2} - a) - n(\lambda_0 - \lambda)) \right\rfloor \\ & \geq \left\lfloor \tfrac{2\sqrt{n}}{n(1-2\alpha)}(\lambda_0(\tfrac{n}{2} - \alpha n) - n(\lambda_0 - \lambda))) \right\rfloor \\ & \geq \left\lfloor \tfrac{2\sqrt{n}}{(1-2\alpha)}(\lambda_0(\tfrac{1}{2} - \alpha) - (\lambda_0 - \lambda)) \right\rfloor = \Omega(\sqrt{n}). \end{aligned}$$

▮

Theorem 1.8 *For all λ, $0 < \lambda < 1$, there exists $n_0 \in \mathsf{N}$ such that for all $n \geq n_0$, $sz(n) \leq \lambda n$.*

Proof:

Let $S(\lambda)$ be the statement

there exists n_0 such that, for all $n \geq n_0$, $sz(n) \leq \lambda n$.

It is a trivial exercise to show that $S(0.7)$ is true.

Let

$$C = \{\lambda \mid S(\lambda)\}.$$

C is closed upwards. Since $0.7 \in C$ we know $C \neq \emptyset$. Assume, by way of contradiction, that $C \neq (0, 1)$. Then there exists $\lambda < \lambda_0$ such that $\lambda \notin C$ and $\lambda_0 \in C$. We can take $\lambda_0 - \lambda$ to be as small as we like. Let n_0 be such that $S(\lambda_0)$ is true via n_0. Let $n \geq n_0$ and let $A \subseteq [n]$ such that $|A| \geq \lambda n$ but A is 3-free. At the end we will fix values for the parameters that (a) allow the proof to go through, and (b) imply $|A| < \lambda n$, a contradiction.

PLAN : We will obtain a $T \subseteq \overline{A}$ that will help us. We will soon see what properties T needs to help us. Consider the bit string in $\{0, 1\}^n$ that represents $T \subseteq [n]$. Say its first 30 bits looks like this:

$T(0)T(1)T(2)T(3) \cdots T(29) = 000111111100001110010111100000$

The set A lives in the blocks of 0's of T (henceforth 0-blocks). We will bound $|A|$ by looking at A on the 'small' and on the 'large' 0-blocks of T. Assume there are t 1-blocks. Then there are $t+1$ 0-blocks. We call a 0-block *small* if it has $< n_0$ elements, and *big* otherwise. Assume there are t^{small} small 0-blocks and t^{big} big 0-blocks. Note that $t^{\text{small}} + t^{\text{big}} = t + 1$ so $t^{\text{small}}, t^{\text{big}} \leq t + 1$. Let the small 0-blocks be $B_1^{\text{small}}, \ldots, B_{t^{\text{small}}}^{\text{small}}$, let their union be B^{small}, let the big 0-blocks be $B_1^{\text{big}}, \ldots, B_{t^{\text{big}}}^{\text{big}}$, and let their union be B^{big}. It is easy to see that

$$|A \cap B^{\text{small}}| \leq t^{\text{small}} n_0 \leq (t+1)n_0.$$

Since each B_i^{big} is bigger than n_0 we must have, for all i, $|A \cap B_i^{\text{big}}| < \lambda_0 |B_i^{\text{big}}|$ (else $A \cap B_i^{\text{big}}$ has a 3-AP and hence A does). It is easy to see that

$$|A \cap B^{\text{big}}| = \sum_{i=1}^{t^{\text{big}}} |A \cap B_i^{\text{big}}| \leq \sum_{i=1}^{t^{\text{big}}} \lambda_0 |B_i^{\text{big}}| \leq \lambda_0 \sum_{i=1}^{t^{\text{big}}} |B_i^{\text{big}}| \leq \lambda_0 (n - |T|).$$

Since A can only live in the (big and small) 0-blocks of T we have

$$|A| = |A \cap B^{\text{small}}| + |A \cap B^{\text{big}}| \leq (t+1)n_0 + \lambda_0(n - |T|).$$

In order to use this inequality to bound $|A|$ we will need T to be big and t to be small, so we want T to be a big set that has few blocks.

If only it was that simple. Actually we can now reveal the

REAL PLAN: The real plan is similar to the easy version given above. We obtain a set $T \subseteq \overline{A}$ and *a parameter d*. A *1-block* is a maximal AP with difference d that is contained in T (that is, if $FIRST$ and $LAST$ are the first and last elements of the 1-block then $FIRST - d \notin T$ and

$LAST + d \notin T$). A *0-block* is a maximal AP with difference d that is contained in $\overline{T}$. Partition T into 1-blocks. Assume there are t of them.

Let $[n]$ be partitioned into $N^0 \cup \cdots \cup N^{d-1}$ where $N_j = \{x \mid x \le n \wedge x \equiv j \pmod{d}\}$.

Fix j, $0 \le j \le d-1$. Consider the bit string in $\{0,1\}^{\lfloor n/d \rfloor}$ that represents $T \cap N_j$ Say the first 30 bits of $T \cap N_j$ look like

$$T(j)T(d+j)T(2d+j)T(3d+j)\cdots T(29d+j) = 000111111100001110010111111100$$

During PLAN we had an intuitive notion of what a 0-block or 1-block was. Note that if we restrict to N_j then that intuitive notion is still valid. For example the first block of 1's in the above example represents $T(3d+j)$, $T(4d+j)$, $T(5d+j)$, $\ldots$, $T(9d+j)$ which is a 1-block as defined formally.

Each 1-block is contained in a particular N_j. Let t_j be the number of 1-blocks that are contained in N_j. Note that $\sum_{j=0}^{d-1} t_j = t$. The number of 0-blocks that are in N_j is at most $t_j + 1$.

Let j be such that $0 \le j \le d-1$. By reasoning similar to that in the above PLAN we obtain

$$|A \cap N_j| \le (t_j + 1)n_0 + \lambda_0(N_j - |T|).$$

We sum both sides over all $j = 0$ to $d-1$ to obtain

$$|A| \le (t+d)n_0 + \lambda_0(n - |T|)$$

In order to use this inequality to bound $|A|$ we need T to be big and t, d to be small. Hence we want a big set T which when looked at mod d, for some small d, decomposes into a small number of blocks.

What is a 1-block within N_j? For example, lets look at $d = 3$ and the bits sequence for T is

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17;
0	0	1	1	1	1	0	1	1	0	1	1	1	1	0	0	0.

Note that T looked at on $N_2 \cup T$ has bit sequence

2	5	8	11	14	17;
0	1	1	1	1	0.

The numbers $5, 8, 11, 14$ are all in T and form a 1-block in the N_2 part. Note that they also from an arithmetic progression with spacing $d = 3$. Also note that this is a maximal arithmetic progression with spacing $d = 3$ since $0 \notin T$ and $17 \notin T$. More generally *1-blocks of T within N_j are maximal arithmetic progressions with spacing d.* With that in mind we can restate the kind of set T that we want.

We want a set $T \subseteq \overline{A}$ and a parameter d such that

1. T is big (so that $\lambda_0(n - |T|)$ is small),
2. d is small (see next item), and
3. the number of maximal arithmetic progressions of length d within T, which is the parameter t above, is small (so that $(t+d)n_0$ is small).

How do we obtain a big subset of $\overline{A}$? We will obtain many pairs $x, y \in A$ such that $2y - x \le n$. Note that since $x, y, 2y - x$ is a 3-AP and $x, y \in A$ we must have $2y - x \in \overline{A}$.

Let α, $0 < \alpha < \frac{1}{2}$, be a parameter to be determined later. (For those keeping track, the parameters to be determined later are now λ_0, λ, n, and α. The parameter n_0 depends on λ_0 so is not included in this list.)

We want to apply Lemma 1.7.2.b to $n, n_0, a = \alpha n$. Hence we need the following conditions.

$$\begin{aligned} \alpha n &\ge n_0 \\ \tfrac{n}{2} &\ge n_0 \\ \tfrac{n}{2} - \alpha n &\ge \sqrt{n} \end{aligned}$$

Assuming these conditions hold, we proceed. By Lemma 1.7.b there is an interval $I \subseteq [\alpha n, \frac{n}{2}]$ of length $O(\sqrt{n})$ such that

$$|A \cap I| \ge \left\lfloor \frac{2\sqrt{n}}{(1 - 2\alpha)} (\lambda_0 (\frac{1}{2} - \alpha) - (\lambda_0 - \lambda)) \right\rfloor = \Omega(\sqrt{n}).$$

By Lemma 1.4 there is a cube $C(e, d_1, \ldots, d_k)$ contained in $|A \cap I|$ with $k = \Omega(\log\log |A \cap I|) = \Omega(\log\log \sqrt{n}) = \Omega(\log\log n)$ and $d \ge \sqrt{n}$.

For i such that $1 \le i \le k$ we define the following.

1. Define $C_0 = \{e\}$ and, for $1 \le i \le k$, define $C_i = C(e, d_1, \ldots, d_i)$.
2. T_i is the third terms of AP's with the first term in $A \cap [1, e-1]$ and the second term in C_i. Formally $T_i = \{2m - x \mid x \in A \cap [1, e-1] \wedge m \in C_i\}$.

Note that, for all i, $T_i \cap A = \emptyset$. Hence we look for a large T_i that can be decomposed into a small number of blocks. We will end up using $d = 2d_{i+1}$.

Note that $T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k$. Hence to obtain a large T_i it suffices to show that T_0 is large and then any of the T_i will be large (though not necessarily consist of a small number of blocks).

Since $C_0 = \{e\}$ we have
$T_0 = \{2m - x \mid x \in A \cap [1, e-1] \wedge m \in C_0\} = \{2e - x \mid x \in A \cap [1, e-1]\}$.

Clearly there is a bijection from $A \cap [1, e-1]$ to T_0, hence $|T_0| = |A \cap [1, e-1]|$. Since $e \in [\alpha n, \frac{n}{2}]$ we have $|A \cap [1, e]| \ge |A \cap [1, \alpha n]|$.

We want to use Lemma 1.6.2 on $A \cap [1, \alpha n]$. Hence we need the condition

$$n - \alpha n \ge n_0.$$

By Lemma 1.6

$$|T_0| \ge |A \cap [1, \alpha n]| \ge \lambda_0 \alpha n - n(\lambda_0 - \lambda) = n(\lambda_0 \alpha - (\lambda_0 - \lambda)).$$

In order for this to be useful we need the following condition

$$\begin{aligned} \lambda - \lambda_0 + \lambda_0 \alpha &> 0 \\ \lambda_0 \alpha &> \lambda_0 - \lambda \end{aligned}$$

We now show that some T_i has a small number of blocks. Since $|T_k| \le n$ (a rather generous estimate) there must exist an i such that $|T_{i+1} - T_i| \le \frac{n}{k}$. Let $t = \frac{n}{k}$ (t will end up bounding the number of 1-blocks).

Partition T_i into maximal AP's with difference $2d_{i+1}$. We call these maximal AP's 1-blocks. We will show that there are $\le t$ 1-blocks by showing a bijection between the blocks and $T_{i+1} - T_i$.

If $z \in T_i$ then $z = 2m - x$ where $x \in A \cap [1, \alpha n - 1]$ and $m \in C_i$. By the definitions of C_i and C_{i+1} we know $m + d_{i+1} \in C_{i+1}$. Hence $2(m + d_{i+1}) - x \in T_{i+1}$. Note that $2(m + d_{i+1}) - x = z + 2d_{i+1}$. In short we have

$$z \in T_i \Rightarrow z + 2d_{i+1} \in T_{i+1}.$$

NEED PICTURE

We can now state the bijection. Let $z_1, \ldots, z_m$ be a block in T_i. We know that $z_m + 2d_{i+1} \notin T_i$ since if it was the block would have been extended to include it. However, since $z_m \in T_i$ we know $z_m + 2d_{i+1} \in T_{i+1}$. Hence $z_m + 2d_{i+1} \in T_{i+1} - T_i$. This is the bijection: map a block to what would be the next element if it was extended. This is clearly a bijection. Hence the number of 1-blocks is at most $t = |T_{i+1} - T_i| \le n/k$.

To recap, we have

$$|A| \le (t + d)n_0 + \lambda_0(n - |T|)$$

with $t \le \frac{n}{k} = O(\frac{n}{\log\log n})$, $d = O(\sqrt{n})$, and $|T| \ge n(\lambda_0\alpha - (\lambda_0 - \lambda))$. Hence we have

$$|A| \le O((\frac{n}{\log\log n} + \sqrt{n})n_0) + n\lambda_0(1 - \lambda + \lambda_0 - \lambda_0\alpha).$$

We want this to be $< \lambda n$. The term $O((\frac{n}{\log\log n} + \sqrt{n})n_0)$ can be ignored since for n large enough this is less than any fraction of n. For the second term we need

$$\lambda_0(1 - \lambda + \lambda_0 - \lambda_0\alpha) < \lambda$$

We now gather together all of the conditions and see how to satisfy them all at the same time.

$$\begin{aligned} \alpha n &\ge n_0 \\ \frac{n}{2} &\ge n_0 \\ \frac{n}{2} - \alpha n &\ge \sqrt{n} \\ n - \alpha n &\ge n_0 \\ \lambda_0\alpha &> \lambda_0 - \lambda \\ \lambda_0(1 - \lambda + \lambda_0 - \lambda_0\alpha) &< \lambda \end{aligned}$$

We first choose λ and λ_0 such that $\lambda_0 - \lambda < 10^{-1}\lambda_0^2$. This is possible by first picking an initial (λ', λ_0') pair and then picking (λ, λ_0) such that $\lambda' < \lambda < \lambda_0 < \lambda_0'$ and $\lambda_0 - \lambda < 10^{-1}(\lambda')^2 < 10^{-1}\lambda_0^2$. The choice of λ_0 determines n_0. We then chose $\alpha = 10^{-1}$. The last two conditions are satisfied:

$\lambda_0\alpha > \lambda_0 - \lambda$ becomes

$$\begin{aligned} 10^{-1}\lambda_0 &> 10^{-1}\lambda_0^2 \\ 1 &> \lambda_0 \end{aligned}$$

which is clearly true.

$\lambda_0(1-\lambda+\lambda_0-\lambda_0\alpha)<\lambda$ becomes

$$\begin{aligned}
\lambda_0(1-10^{-1}\lambda_0^2-10^{-1}\lambda_0) &< \lambda \\
\lambda_0-10^{-1}\lambda_0^3-10^{-1}\lambda_0^2 &< \lambda \\
\lambda_0-\lambda-10^{-1}\lambda_0^3-10^{-1}\lambda_0^2 &< 0 \\
10^{-1}\lambda_0^2-10^{-1}\lambda_0^3-10^{-1}\lambda_0^2 &< 0 \\
-10^{-1}\lambda_0^3 &< 0
\end{aligned}$$

which is clearly true.

Once λ, λ_0, n_0 are picked, you can easily pick n large enough to make the other inequalities hold. ▮

2 What more is known?

The following is known.

Theorem 2.1 *For every $\lambda > 0$ there exists n_0 such that for all $n \ge n_0$, $sz(n) \le \lambda n$.*

This has been improved by Heath-Brown [3] and Szemeredi [8]

Theorem 2.2 *There exists c such that $sz(n) = \Omega(n\frac{1}{(\log n)^c})$. (Szemeredi estimates $c \le 1/20$).*

Bourgain [1] improved this further to obtain the following.

Theorem 2.3 $sz(n) = \Omega(n\sqrt{\frac{\log\log n}{\log n}})$.

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