Beck's Surplus Tic Tac Toe Game
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1 Introduction

Consider the following game:

Two players Mark (for Maker) and Betty (for Breaker) alternate (Mark going first) placing M's and B's on an $n \times n$ checkerboard. Mark wins if he can get n M's in either the same row or the same column (getting n on the diagonal does not give him a win). Betty wins if she prevents him from doing this.

The above game is stupid.

Exercise 1 Show that for $n \geq 3$ Betty wins the above game.

Mark cannot win this game. But what if we lower our expectations? Consider the following game

Definition 1.1 Let $f : \mathsf{N} \to \mathsf{N}$. The *tic-tac-toe-$f(n)$-surplus game (henceforth ttt-$f(n)$ game)* is as follows. Two players Mark (for Maker) and Betty (for Breaker) alternate (Mark going first) placing M's and B's on an $n \times n$ checkerboard. Mark wins if he can get $\frac{n}{2} + f(n)$ in either the same row or the same column (getting $\frac{n}{2} + f(n)$ on the diagonal does not give him a win). Betty wins if she prevents him from doing this.

Question: For what value of $f(n)$ does Mark have a winning strategy? For what value of $f(n)$ does Betty have a winning strategy?

We show that $\Omega(\sqrt{n}) \leq f(n) \leq O(\sqrt{n \log n})$. This is a result of Beck from [1]; however, we give a self contained proof. In addition, our exposition is online and hence available to anyone.

Definition 1.2 RC is the set of rows and column. We assume that RC is ordered so that one can refer to *the least element of RC such that ...*.

2 Mark Can Achieve Surplus $\Omega(\sqrt{n})$

Theorem 2.1 *There exists constants $c \in \mathsf{R}^+$ and $n_0 \in \mathsf{N}$ such that, for all $n \geq n_0$, there is a strategy by which Mark can win the ttt-$c\sqrt{n}$ game.*

Proof:

We define a potential function which will measure how well Mark is doing. Mark's strategy will be to (essentially)increase its value as much as possible.

Let $0 < \epsilon < 1$ be a parameter to be named later (it will be $\Theta\left(\frac{1}{\sqrt{n}}\right)$).

A *turn* is a pair of moves- one by Mark and the response by Betty. We will assume that n is even to avoid half-turns (Mark goes and there is no response from Betty since the game is over).

Let t be how many turns have already been made. Let $M_t(A)$ be how many M's are in A after t turns. Let $B_t(A)$ be how many B's are in A after t turns. We define the potential function:

$$\Phi_t = \sum_{A \in RC} (1+\epsilon)^{M_t(A)}(1-\epsilon)^{B_t(A)}.$$

Here is the strategy for M. Assume that t turns have already occurred (t could be 0).

Strategy for Mark: There are two possibilities.

1. There is some $A \in RC$ such that $M_t(A) - B_t(A) \geq 2c\sqrt{n}$. Let A be such that $M_t(A) - B_t(A)$ is maximized (if this A is non unique take the least such one). Place an M in A.

2. There is no such A. Play on an element of RC such that $\Phi_{t+1} - \Phi_t$ is maximized. (If there is a tie then use the least such element of RC.)

Assume Mark has played this strategy. There are two cases; however, we will show that Case 2 does not occur.

Case 1: There is a stage t such that the first possibility of the strategy occurs. Let t_0 be the least such t. Let A be the element of RC that Mark places an M in during turn t. It is easy to see that Mark will play in A for the rest of the game. It is also easy to see that

$$M_{n^2/2}(A) - B_{n^2/2}(A) \geq 2c\sqrt{n}.$$

Since

$$M_{n^2/2}(A) + B_{n^2/2}(A) = n$$

We have

$$M_{n^2/2}(A) \geq \frac{n}{2} + c\sqrt{n}.$$

Case 2: There is no such stage t. Let Δ be defined as

$$\Delta = \max_{t,A} \frac{M_t(A) - B_t(A)}{2}.$$

Note that, for all t, for all A,

$$M_t(A) - B_t(A) \leq 2\Delta.$$

We find a lower bound on $\Phi_{n^2/2}$ based on Δ. We will then find an (easy) upper bound on $\Phi_{n^2/2}$. We will use this upper and lower bound to get a lower bound on Δ.

The potential will decrease over time, but we need to show that it does not decrease too much. Let $t+1 \leq \frac{n^2}{2}$. How big can $\Phi_{t+1} - \Phi_t$ be?

We will need the following fact:

Fact 1:

1. If Mark puts the M on the intersection of row A_1 and column A_2 the potential function goes up by
$$\epsilon((1+\epsilon)^{M_t(A_1)}(1-\epsilon)^{B_t(A_1)} + (1+\epsilon)^{M_t(A_2)}(1-\epsilon)^{B_t(A_2)})$$

2. If Mark puts the M on the intersection of row A_1 and column A_2, and then Betty puts the B on the intersection of row A_3 and column A_4, $A_1 \neq A_3$, $A_2 \neq A_4$ then

$$(1+\epsilon)^{M_t(A_1)}(1-\epsilon)^{B_t(A_1)}+(1+\epsilon)^{M_t(A_2)}(1-\epsilon)^{B_t(A_2)}) \geq (1+\epsilon)^{M_t(A_3)}(1-\epsilon)^{B_t(A_3)}+(1+\epsilon)^{M_t(A_4)}(1-\epsilon)^{B_t(A_4)})$$

3. If Mark puts the M on the intersection of row A_1 and column A_2, and then Betty puts the B on the intersection of row A_1 and column A_4, $A_2 \neq A_4$ then

$$+\left(\epsilon(1+\epsilon)^{M_t(A_2)}(1-\epsilon)^{B_t(A_2)} - \epsilon(1+\epsilon)^{M_t(A_4)}(1-\epsilon)^{B_t(A_4)}\right)$$

Proof of Fact 1:
1)
Marks move only affects the potential on row A_1 and column A_2. The potential goes up by

$$(1+\epsilon)^{M_t(A_1)+1}(1-\epsilon)^{B_t(A_1)}+(1+\epsilon)^{M_t(A_2)+1}(1-\epsilon)^{B_t(A_2)}-(1+\epsilon)^{M_t(A_1)}(1-\epsilon)^{B_t(A_1)}-(1+\epsilon)^{M_t(A_2)}(1-\epsilon)^{B_t(A_2)}$$

$=$

$$(1+\epsilon)^{M_t(A_1)}(1-\epsilon)^{B_t(A_1)}(1+\epsilon-1)+(1+\epsilon)^{M_t(A_2)}(1-\epsilon)^{B_t(A_2)}(1+\epsilon-1)$$

$=$

$$\epsilon(1+\epsilon)^{M_t(A_1)}(1-\epsilon)^{B_t(A_1)})+\epsilon(1+\epsilon)^{M_t(A_2)}(1-\epsilon)^{B_t(A_2)})$$

$=$

$$\epsilon((1+\epsilon)^{M_t(A_1)}(1-\epsilon)^{B_t(A_1)}+(1+\epsilon)^{M_t(A_2)}(1-\epsilon)^{B_t(A_2)})$$

2) Since Mark's move maximizes potential it must create a bigger change of potential then the move that puts a marker at the intersection of row A_3 and column A_4. The inequality follows from this observation and Item 1.

3) This is a calculation similar to items 1 and 2 above.
End of Proof of Fact 1

Case 1: $A_1 \neq A_4$ and $A_2 \neq A_4$. We look at $\Phi_t - \Phi_{t+1}$. We need only look at the parts of the sum that involve A_1, A_2, A_3, A_4

$\Phi_{t+1} - \Phi_t =$

$$(1+\epsilon)^{M_t(A_1)+1}(1-\epsilon)^{B_t(A_1)} - (1+\epsilon)^{M_t(A_1)}(1-\epsilon)^{B_t(A_1)}$$

$+$

$$(1+\epsilon)^{M_t(A_2)+1}(1-\epsilon)^{B_t(A_2)} - (1+\epsilon)^{M_t(A_2)}(1-\epsilon)^{B_t(A_2)}$$

+

$$(1+\epsilon)^{M_t(A_3)}(1-\epsilon)^{B_t(A_3)+1} - (1+\epsilon)^{M_t(A_3)}(1-\epsilon)^{B_t(A_3)}$$

+

$$(1+\epsilon)^{M_t(A_4)}(1-\epsilon)^{B_t(A_4)+1} - (1+\epsilon)^{M_t(A_4)}(1-\epsilon)^{B_t(A_4)}$$

=

$$(1+\epsilon)^{M_t(A_1)}(1-\epsilon)^{B_t(A_1)}(1+\epsilon-1)$$

+

$$(1+\epsilon)^{M_t(A_2)}(1-\epsilon)^{B_t(A_2)}(1+\epsilon-1)$$

+

$$(1+\epsilon)^{M_t(A_3)}(1-\epsilon)^{B_t(A_3)}(1-\epsilon-1)$$

+

$$(1+\epsilon)^{M_t(A_4)}(1-\epsilon)^{B_t(A_4)}(1-\epsilon-1)$$

=

$$\epsilon(1+\epsilon)^{M_t(A_1)}(1-\epsilon)^{B_t(A_1)}$$

+

$$\epsilon(1+\epsilon)^{M_t(A_2)}(1-\epsilon)^{B_t(A_2)}$$

$$-\epsilon(1+\epsilon)^{M_t(A_3)}(1-\epsilon)^{B_t(A_3)}$$

$$-\epsilon(1+\epsilon)^{M_t(A_4)}(1-\epsilon)^{B_t(A_4)}$$

$$= \epsilon\Big((1+\epsilon)^{M_t(A_1)}(1-\epsilon)^{B_t(A_1)} + \epsilon(1+\epsilon)^{M_t(A_2)}(1-\epsilon)^{B_t(A_2})$$

$$-(1+\epsilon)^{M_t(A_3)}(1-\epsilon)^{B_t(A_3)} - (1+\epsilon)^{M_t(A_4)}(1-\epsilon)^{B_t(A_4)})\Big)$$

This quantity is ≥ 0 by Fact 1.

Case 2: $A_1 = A_3$. Only rows A_1, A_3 and column A_4 are affected by the turn.

$\Phi_{t+1} - \Phi_t =$

$$(1+\epsilon)^{M_t(A_1)+1}(1-\epsilon)^{B_t(A_1)+1} - (1+\epsilon)^{M_t(A_1)}(1-\epsilon)^{B_t(A_1)}$$

+

$$(1+\epsilon)^{M_t(A_2)+1}(1-\epsilon)^{B_t(A_2)} - (1+\epsilon)^{M_t(A_2)}(1-\epsilon)^{B_t(A_2)}$$

$+$

$$(1+\epsilon)^{M_t(A_4)}(1-\epsilon)^{B_t(A_4)+1} - (1+\epsilon)^{M_t(A_4)}(1-\epsilon)^{B_t(A_4)}$$

$=$

$$(1+\epsilon)^{M_t(A_1)}(1-\epsilon)^{B_t(A_1)}((1+\epsilon)(1-\epsilon)-1)$$

$+$

$$(1+\epsilon)^{M_t(A_2)}(1-\epsilon)^{B_t(A_2)}(1+\epsilon-1)$$

$+$

$$(1+\epsilon)^{M_t(A_4)}(1-\epsilon)^{B_t(A_4)}(1-\epsilon-1)$$

$=$

$$-\epsilon^2(1+\epsilon)^{M_t(A_1)}(1-\epsilon)^{B_t(A_1)}$$

$$+\left(\epsilon(1+\epsilon)^{M_t(A_2)}(1-\epsilon)^{B_t(A_2)} - \epsilon(1+\epsilon)^{M_t(A_4)}(1-\epsilon)^{B_t(A_4)}\right)$$

The expression in big parenthesis must be ≥ 0 by Fact 1.2.
So we have

$$\Phi_{t+1} - \Phi_t \geq -\epsilon^2(1+\epsilon)^{M_t(A_1)}(1-\epsilon)^{B_t(A_1)}$$

$$\Phi_{t+1} \geq \Phi_t - \epsilon^2(1+\epsilon)^{M_t(A_1)}(1-\epsilon)^{B_t(A_1)}$$

We want to get this in terms of Δ. By the definition of Δ

$$M_t(A_1) - B_t(A_1) \leq 2\Delta$$

Let $Z_t = \frac{M_t(A_1)+B_t(A_1)}{2}$. Then

$$\begin{array}{rl} M(A_1) & \leq Z_t + \Delta \\ B(A_1) & \geq Z_t - \Delta \end{array}$$

Hence

$$\Phi_{t+1} \quad \geq \Phi_t - \epsilon^2(1+\epsilon)^{Z_t+\Delta}(1-\epsilon)^{Z_t-\Delta} \geq \Phi_t - \epsilon^2\left(\frac{1+\epsilon}{1-\epsilon}\right)^{\Delta}(1-\epsilon^2)^{Z_t} \geq \Phi_t - \epsilon^2\left(\frac{1+\epsilon}{1-\epsilon}\right)^{\Delta}$$

We use an approximation to simplify this expression. Note that

$$\frac{1+\epsilon}{1-\epsilon} = 1 + \frac{2\epsilon}{1-\epsilon} \sim e^{2\epsilon}.$$

Hence we have

$$\Phi_{t+1} \geq \Phi_t - \epsilon^2 e^{2\epsilon\Delta}$$

$$\Phi_0 = \sum_{A \in RC} (1+\epsilon)^{M_0(A)}(1-\epsilon)^{B_0(A)} = \sum_{A \in RC} (1+\epsilon)^0 (1-\epsilon)^0 = 2n.$$

Hence

$$\Phi_{n^2/2} \geq 2n - \epsilon^2 e^{2\epsilon\Delta} \frac{n^2}{2} = 2n - e^{2\epsilon\Delta} \frac{\epsilon^2 n^2}{2}.$$

We will now pick ϵ though it will be in terms of another constant. Let $\epsilon = \sqrt{2\beta/n}$ where β will be chosen later.

Hence we have

$$\Phi_{n^2/2} \geq 2n - e^{\Delta\sqrt{8\beta/n}} \beta n.$$

We also have an upper bound on $\Phi_{n^2/2}$.

$$\Phi_{n^2/2} = \sum_{A \in RC} (1+\epsilon)^{M_{n^2/2}(A)} (1-\epsilon)^{B_{n^2/2}(A)}$$

By the definition of Δ

$$\begin{array}{ll} M_{n^2/2}(A) & \leq \frac{n}{2} + \Delta \\ B_{n^2/2}(A) & \geq \frac{n}{2} - \Delta \end{array}$$

Hence

$$\begin{array}{ll} \Phi_{n^2/2} & = \sum_{A \in RC} (1+\epsilon)^{M_{n^2/2}(A)} (1-\epsilon)^{B_{n^2/2}(A)} \leq \sum_{A \in RC} (1+\epsilon)^{n/2+\Delta} (1-\epsilon)^{n/2-\Delta} \\ & \leq 2n(1+\epsilon)^{n/2+\Delta}(1-\epsilon)^{n/2-\Delta} \\ & \leq 2n \left(\frac{1+\epsilon}{1-\epsilon}\right)^{\Delta} (1-\epsilon^2)^{n/2} \end{array}$$

We use two approximations to simplify this expression.

From above we have

$$\frac{1+\epsilon}{1-\epsilon} = 1 + \frac{2\epsilon}{1-\epsilon} \sim e^{2\epsilon}.$$

We also use

$$1 - \epsilon^2 = e^{-\epsilon^2}$$

Hence we have

$$\Phi_{n^2/2} \leq 2ne^{2\epsilon\Delta}e^{-\epsilon^2 n/2}$$

We now use the definition of ϵ to obtain

$$\Phi_{n^2/2} \leq 2ne^{\Delta\sqrt{8\beta/n}}e^{-\beta}$$

Combining the upper and lower bounds on $\Phi_{n^2/2}$ we obtain the following.

$$2n - e^{\Delta\sqrt{8\beta/n}}\beta n \leq 2ne^{\Delta\sqrt{8\beta/n}}e^{-\beta}$$

$$2 - e^{\Delta\sqrt{8\beta/n}}\beta \leq 2e^{\Delta\sqrt{8\beta/n}}e^{-\beta}$$

$$2 \leq e^{\Delta\sqrt{8\beta/n}}\beta + 2e^{\Delta\sqrt{8\beta/n}}e^{-\beta}$$

$$2 \leq e^{\Delta\sqrt{8\beta/n}}(\beta + 2e^{-\beta})$$

$$\frac{2}{\beta + 2e^{-\beta}} \leq e^{\Delta\sqrt{\frac{8\beta}{n}}}$$

$$\ln\left(\frac{2}{\beta + 2e^{-\beta}}\right) \leq \Delta\sqrt{\frac{8\beta}{n}}$$

$$\Delta \geq \sqrt{\frac{n}{8\beta}}\ln\left(\frac{2}{\beta + 2e^{-\beta}}\right)$$

$$\Delta \geq \sqrt{\frac{1}{8\beta}}\ln\left(\frac{2}{\beta + 2e^{-\beta}}\right)\sqrt{n}$$

Pick β so that the constant in front of the $\sqrt{n}$ is positive. The reader may want to pick β so as to maximize the constant.

Let d be such that $\Delta \geq d\sqrt{n}$. By the definition of Δ there exists a t and an A such that

$$M_t(A) - B_t(A) = 2\Delta \geq 2d\sqrt{n}.$$

Let $c = \frac{d}{2}$. Hence

$$M_t(A) - B_t(A) \geq c\sqrt{n}.$$

At this stage t the first part of the strategy will happen. Hence this case, case 2, cannot occur.

$\blacksquare$

3 Betty Can Make Sure Surplus $\le O(\sqrt{n \ln n})$

(This section will only sketch the proof.)

Theorem 3.1 *There exist constants $c \in \mathsf{R}^+$ and $n \in \mathsf{N}$ such that, for all $n \ge n_0$, there is a strategy by which Betty can win the ttt-$c\sqrt{n \ln n}$ game.*

Proof:

We let Betty go first in this game. This will only affect the constants in the asymptotics.

We define a potential function which will measure how well Mark is doing. Betty's strategy will be to decrease its value as much as possible.

Let $0 < \epsilon < 1$ be a parameter to be named later (it will be $\Theta\left(\sqrt{\frac{\ln n}{n}}\right)$).

A *turn* is a pair of moves- one by Betty and the response by Mark. We will assume that n is even to avoid half-turns (Betty goes and there is no response from Mark since the game is over).

Let t be how many turns have already been made. Let $M_t(A)$ be how many M's are in A after t turns. Let $B_t(A)$ be how many B's are in A after t turns. We define the potential function:

$$\Phi_t = \sum_{A \in RC} (1+\epsilon)^{M_t(A)-((1+\epsilon)n/2)}(1-\epsilon)^{B_t(A)-((1-\epsilon)n/2)}.$$

Strategy for Betty: Assume that t turns have already occurred (t could be 0). Play on an element of RC such that $\Phi_t - \Phi_{t+1}$ is maximized. (If there is a tie then use the least such element of RC.)

Claim 1: If Betty plays the strategy above then the potential always either stays the same or decreases.

Proof of Claim 1:

This is a calculation that we will omit. (This is what I meant when I said we would sketch the proof)

End of Proof of Claim 1

Using $(1+\epsilon) \sim e^{\epsilon}$ and $(1-\epsilon) \sim e^{-\epsilon}$ we obtain the following.

$$\begin{aligned}\Phi_0 = \textstyle\sum_{A \in RC}(1+\epsilon)^{-((1+\epsilon)n/2)}(1-\epsilon)^{-((1-\epsilon)n/2)} &= 2n(1+\epsilon)^{-((1+\epsilon)n/2)}(1-\epsilon)^{-((1-\epsilon)n/2)} \\ &\sim 2ne^{-\epsilon^2 n}\end{aligned}$$

We will now set ϵ though it will depend on a constant. Let $\epsilon = \sqrt{\frac{\beta \ln n}{n}}$. Note that

$$2ne^{-\epsilon^2 n} = 2ne^{-\beta \ln n} = 2n \times n^{-\beta} = 2n^{1-\beta}.$$

For the next few equations let $M(A) = M_{n^2/2}(A)$ and $B(A) = B_{n^2/2}(A)$.

Recall that

$$\Phi_{n^2/2} = \sum_{A \in RC} (1+\epsilon)^{M(A)-((1+\epsilon)n/2)}(1-\epsilon)^{B(A)-((1-\epsilon)n/2)}.$$

Also recall that $\Phi_{n^2/2} \le \Phi_0$.

If there is an A such that

$$M(A) \geq \frac{n}{2} + \frac{1+\epsilon}{2} = \frac{n}{2} + \Theta\left(\sqrt{\frac{\log n}{n}}\right)$$

that summand will be greater than 1. Hence if $\Phi_{n^2/2} < 1$ then there can be no such A. Since $\Phi_{n^2/2} \leq \Phi_0$ all we need is $\Phi_0 < 1$. We can accomplish that by taking $\beta < 1$. ∎

4 This section has a proof of something I know is false- Help me if you can

When I was trying to derive what ϵ should be I got results that did not make sense. It may be an arithmetic mistake or I may have a fundamental misunderstanding of something. If you can tell me whats wrong I will be enlightened.

Assume Betty has played this strategy. Let Δ be defined as

$$\Delta = \max_{A \in RC} M_{n^2/2}(A) - B_{n^2/2}(A).$$

We use the approximation $(1+x) \sim e^x$ on $\Phi_{n^2/2}$.

$$\begin{array}{ll} \Phi_{n^2/2} & = \sum_{A \in RC} (1+\epsilon)^{M(A)-((1+\epsilon)n/2)} (1-\epsilon)^{B(A)-((1-\epsilon)n/2)} \\ & \sim \sum_{A \in RC} e^{\epsilon(M(A)-((1+\epsilon)n/2))} e^{-\epsilon(B(A)-((1-\epsilon)n/2))} \\ & \sim \sum_{A \in RC} e^{\epsilon(M(A)-B(A))-\epsilon^2 n} \end{array}$$

Let A be the row where $M(A) = B(A) = 2\Delta$. The sum above is greater than one of its summands. Hence

$$\Phi_{n^2/2} \geq e^{2\epsilon\Delta - \epsilon^2 n}$$

Since $\Phi_{n^2/2} \leq \Phi_0$ we have

$$e^{2\epsilon\Delta - \epsilon^2 n} \leq \Phi_{n^2/2} \leq \Phi_0 \leq 2ne^{-\epsilon^2 n}$$

KEY: the $e^{-\epsilon^2 n}$ cancel out. This leads to results that do not make sense.

$$e^{2\epsilon\Delta} \leq 2n$$

$$2\epsilon\Delta \leq \ln(2n)$$

$$\Delta \leq \frac{\ln(2n)}{\epsilon}$$

Gee, I could just take ϵ to be (say) $\frac{1}{\ln(2n)}$ and get that

$$\Delta \leq O((\log n)^2).$$

This contradicts Theorem 2.1.

One possible Fix: I cheated a bit by letting Betty go first. What if Mark goes first? The proof would start the potential after Mark's first move. Φ_0 does start out a little bigger, but this did not help since I still got all of the $e^{-\epsilon^2}$ to cancel out. Also, for large n, the player who goes first shouldn't matter.

Another possible fix: The proof that the potential never increases— did that impose bounds on ϵ. The proof by Beck didn't seem to.

Another possible fix: Do the approximation more carefully. This wouldn't help anything since the approximation I am using is correct for large n.

References

[1] J. Beck. Surplus of graphs and the local Lovasz lemma. In *Building bridges between math and computer science*, pages 47–103, New York, 2008. Springer. Bolyai society mathematical studies number 19.