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Generalized quantifiers and pebble games on finite structures

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Abstract

First-order logic is known to have a severely limited expressive power on finite structures. As a result, several different extensions have been investigated, including fragments of second-order logic, fixpoint logic, and the infinitary logic $\mathscr{L}^{\omega}_{\infty\omega}$ in which every formula has only a finite number of variables. In this paper, we study *generalized quantifiers* in the realm of finite structures and combine them with the infinitary logic $\mathscr{L}^{\omega}_{\infty\omega}$ to obtain the logics $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$, where $\boldsymbol{Q} = \{Q_i\colon i \in I\}$ is a family of generalized quantifiers on finite structures. Using the logics $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$, we can express polynomial-time properties that are not definable in $\mathscr{L}^{\omega}_{\infty\omega}$, such as "there is an even number of x" and "there exists at least $n/2$ x" (n is the size of the universe), without going to second-order logic.

We show that equivalence of finite structures relative to $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$ can be characterized in terms of certain pebble games that are a variant of the *Ehrenfeucht–Fraïssé games*. We combine this game-theoretic characterization with sophisticated combinatorial tools from Ramsey theory, such as van der Waerden's Theorem and Folkman's Theorem, in order to investigate the scope and limits of generalized quantifiers in finite model theory. We obtain sharp lower bounds for expressibility in the logics $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$ and discover an intrinsic difference between adding finitely many simple unary generalized quantifiers to $\mathscr{L}^{\omega}_{\infty\omega}$ and adding infinitely many. In particular, we show that if $\boldsymbol{Q}$ is a finite sequence of simple unary generalized quantifiers, then the equicardinality, or Härtig, quantifier is not definable in $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$. We also show that the query "does the equivalence relation E have an even number of equivalence classes" is not definable in the extension $\mathscr{L}^{\omega}_{\infty\omega}(I, \boldsymbol{Q})$ of $\mathscr{L}^{\omega}_{\infty\omega}$ by the Härtig quantifier I and any finite sequence $\boldsymbol{Q}$ of simple unary generalized quantifiers.

1. Introduction and summary of results

For many decades traditional mathematical logic focused on the study of first-order logic on the class of all structures (both finite and infinite) or on fixed infinite

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structures of mathematical significance. During the late 1950s and the early 1960s researchers initiated an investigation of logics that extend first-order logic. The motivation for this line of research came mainly from the fact that first-order logic has rather limited expressive power on important infinite mathematical structures.

The work of Mostowski [44] on *cardinality quantifiers*, such as *there are infinitely many elements*, was the starting point of the research on extensions of first-order logic. Soon after this, Tarski [45] initiated the study of *infinitary languages*, i.e., extensions of first-order logic in which infinitely long expressions are allowed in the syntax. In his pioneering papers [39, 40], Lindström introduced *generalized quantifiers* and obtained *abstract characterizations* of first-order logic. The above investigations laid the foundation for the systematic study during the 1970s and the 1980s of *extended logics* and for the development of *abstract model theory* as the area of research whose aim is to classify these extended logics and to discover the relations between them (cf. [8]).

The 1970s and the 1980s were also a period of increasing interaction between logic and computer science. While exploring the connections between logic and computer science, researchers realized that the finite structures, and not the infinite structures of classical logic, are the ones that are relevant to computer science (cf. [21, 22]). Out of these considerations, *finite model theory* emerged as an active area of research that has been developing steadily over the past 20 years.

Several individual extensions of first-order logic were investigated in the context of finite structures from the perspectives of expressive power and relations to complexity classes. Fagin [15] characterized the class NP of nondeterministic polynomial-time problems as the set of properties expressible by *existential second-order sentences* on finite structures. Σ^1_1 sentences, or *existential second-order* sentences, are expressions of the form

$$\exists S_1 \dots \exists S_n \varphi(S_1, \dots, S_n, R_1, \dots, R_m),$$

where $S_1, \dots, S_n, R_1, \dots, R_m$ are relational variables and $\varphi(S_1, \dots, S_n, R_1, \dots, R_m)$ is a first-order sentence. Fagin's [15] result shows that even the simplest fragment of second-order logic is too powerful on finite structures. It also raises the question: what has to be added to first-order logic in order to capture exactly all polynomial-time properties on finite structures?

Aho and Ullman [4] pointed out that first-order logic on finite structures has rather limited expressive power (cf. also [16]). Intuitively, these limitations arise from the fact that first-order logic on finite structures lacks a recursion mechanism. To remedy this situation, Chandra and Harel [11] introduced *fixpoint logic* on finite structures, which can be described succinctly as first-order logic augmented with least fixpoints of positive first-order formulas. Fixpoint logic had been studied earlier by logicians on infinite structures under the name *inductive definability* and turned out to be a powerful tool for analyzing second-order quantification on infinite structures (cf. [43, 3]).

The expressive power and the structural properties of fixpoint logic were investigated by several researchers, including [2, 48, 28, 23]. From a computational standpoint, every property expressible in fixpoint logic on finite structures is computable in

PTIME. On the positive side, fixpoint logic not only can express connectivity, acyclicity, and 2-colorability, but it can also capture certain properties that are *complete* for polynomial time, such as the *path systems* query [12]. On the other hand, fixpoint logic is unable to express many properties that directly or indirectly involve counting, such as "there is an even number of elements" or "there is an Eulerian cycle" (cf. [11]).

This deficiency of fixpoint logic can be overcome if the inputs are restricted to be *ordered* finite structures, i.e., if it is assumed that the underlying vocabulary contains a binary symbol $<$ which is always interpreted as a total ordering of the universe of the input structure. Indeed, Immerman [28] and Vardi [48] showed that on ordered finite structures a property is expressible in fixpoint logic if and only if it is computable in polynomial time.

Although the above result identifies the expressive power of fixpoint logic on an important class of finite structures, it does not answer the question of what has to be added to first-order logic in order to capture polynomial time on finite structures. Equivalently, this question can be phrased as: what has to be added to first-order logic to capture exactly all *order-independent* polynomial-time properties of ordered finite structures? (a property of ordered structures is *order-independent* if it does not depend on the actual total ordering on the universe of the structure). These questions have attracted considerable attention in both complexity theory (cf. [28, 22, 30]) and database theory ([2]), because, although an order is always present when representing or encoding finite structures by strings, in practice all algorithmic problems about finite structures have to do with order-independent properties. In spite of considerable efforts, however, so far there has not been found an extension of first-order logic that captures exactly all polynomial-time properties on finite structures. This state of affairs has motivated Gurevich [22] to make the bold conjecture that *no* such logic exists.

To enhance the power of fixpoint logic, Immerman [28] augmented it with *counting quantifiers* $(\exists i x)$, for each positive integer i. The interpretation of the quantifier $(\exists i x)$ is that "there are at least i elements x such that...". Fixpoint logic with counting quantifiers becomes more powerful than fixpoint logic on two-sorted finite structures in which one of the sorts is for the universe of the original structures, while the other is used to do arithmetic on the counting quantifiers. In particular, properties such as "there is an even number of elements" are easily expressible in that setup. Immerman [28] conjectured that fixpoint logic with counting quantifiers can express all polynomial-time properties. This conjecture, however, was refuted later on by Cai et al. [9].

The limited expressive power of first-order logic can also be increased by permitting infinitary rules in the syntax. Note that the well-known infinitary logic $\mathscr{L}_{\omega_1\omega}$, which allows for countable disjunctions and conjunctions, is too potent on finite structures to be of interest, since every class of finite structures is definable in this logic. A better alternative is provided by the family $\mathscr{L}^k_{\infty\omega}$, $k \geqslant 1$, of infinitary logics that allow infinite disjunctions and conjunctions, but have only a total of k distinct variables. These logics were originally introduced by Barwise [7] with infinite structures in mind, but turned out to be of interest and use in finite model theory. Indeed, they underlie much of the work in [27, 29, 9, 30] and have also been studied in their own right ([35, 36]).

On finite structures the union $\mathscr{L}^{\omega}_{\infty\omega} = \bigcup_{k=1}^{\infty} \mathscr{L}^{k}_{\infty\omega}$ of the infinitary logics $\mathscr{L}^{k}_{\infty\omega}$ is a proper extension of fixpoint logic, but it suffers from the same limitations as fixpoint logic when it comes to "counting". Many of these limitations are easily overcome if the counting quantifiers $(\exists i x)$, $i \geqslant 1$, are added to $\mathscr{L}^{k}_{\infty\omega}$. For example, the property "there is an even number of elements" is expressible by the formula:

$$\bigvee_{i \geqslant 1} ((\exists 2i x)(x = x) \wedge \neg (\exists 2i + 1 x)(x = x)).$$

Similarly, the property "there is an Eulerian cycle" is expressible in $\mathscr{L}^{2}_{\infty\omega}(\boldsymbol{C})$ by the sentence

$$(\forall x)\left[\bigvee_{n \geqslant 1} (\exists 2n y)E(x, y) \wedge \neg (\exists 2n + 1 y)E(x, y)\right].$$

The above sentence asserts that every node in a graph $G = (V, E)$ is of even degree, which is a condition equivalent to the existence of an Eulerian cycle in G. Notice that the agreement that $(\exists i x)$ is a *new* quantifier entails that only one variable, namely x, is used in the above formula (in contrast, the first-order translation of each $(\exists i x)$ requires i variables). This is quite important when we are very conscious of the total number of variables used in a formula. We let $\mathscr{L}^{k}_{\infty\omega}(\boldsymbol{C})$ denote the logic obtained by augmenting $\mathscr{L}^{k}_{\infty\omega}$ with all counting quantifiers and we write $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{C})$ for the union $\bigcup_{k=1}^{\infty} \mathscr{L}^{k}_{\infty\omega}(\boldsymbol{C})$. As it turns out, even the logic $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{C})$ cannot express all polynomial-time properties of graphs. Indeed, this is again a consequence of the main result in Cai et al. [9].

There is an alternative approach to enhancing the expressive power of $\mathscr{L}^{\omega}_{\infty\omega}$, an approach which is motivated by the work of Mostowski [44] and Lindström [39, 40] on generalized quantifiers on infinite structures and by the subsequent developments in the study of extended logics [8]. Rather than adding all infinitely many counting quantifiers at once, we may add individual tailor-made *generalized quantifiers* that are meaningful on finite structures. For example, we need not add all counting quantifiers to express "there is an even number of elements", all we need is the quantifier

$$Q_{\text{even}} x \varphi(x) \Leftrightarrow |\{x\colon \varphi(x)\}| \text{ is even.}$$

Similarly, we need but one new quantifier to express "at least half of the elements satisfy ...", or "the number of elements satisfying ... divides the size of the universe". These quantifiers are said to be *of type* (1) or *simple unary generalized quantifiers*, because they apply to only one formula (*simple*) and they bind only one variable (*unary*). Generalized quantifiers on finite structures were studied explicitly for the first time by Hájek [24]. Our goal in this paper is to undertake a systematic study of generalized quantifiers in finite model theory. To this effect, we introduce the family of infinitary logics $\mathscr{L}^{k}_{\infty\omega}(\boldsymbol{Q})$, $k \geqslant 1$, where $\boldsymbol{Q} = \{Q_i\colon i \in I\}$ is a collection (finite or infinite) of simple unary generalized quantifiers on finite structures. These logics are obtained by augmenting in a natural way the infinitary logics $\mathscr{L}^k_{\infty\omega}$ with the quantifiers Q_i, $i \in I$.

We study both the model-theoretic properties and the expressive power of the infinitary logics $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, $k \geqslant 1$, on finite structures. Our first result is that the logics $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, $k \geqslant 1$, are sufficiently well-behaved to admit a game-theoretic characterization of *equivalence* in $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, i.e., we can tell whether or not two finite structures satisfy the same sentences of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ by using appropriate pebble games. These games generalize both the k-pebble games for $\mathscr{L}^k_{\infty\omega}$ in [7, 27] and the counting k-pebble game for $\mathscr{L}^k_{\infty\omega}(\boldsymbol{C})$ in [9, 30]. As the first application of the pebble games for $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, we obtain a structural characterization of the counting quantifiers by showing that the counting quantifiers are essentially the only simple unary generalized quantifiers on finite structures that are *monotone* and possess a certain useful closure property, called *relativization*.

In terms of expressive power, it is easy to see that the infinitary logic $\mathscr{L}^\omega_{\infty\omega}(\boldsymbol{C})$ with the counting quantifiers can subsume every logic $\mathscr{L}^\omega_{\infty\omega}(\boldsymbol{Q}) = \bigcup_{k=1}^{\infty} \mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, where $\boldsymbol{Q} = \{Q_i\colon i \in I\}$ is an arbitrary collection of simple unary generalized quantifiers on finite structures. However, if a formula φ of $\mathscr{L}^\omega_{\infty\omega}(\boldsymbol{C})$ defines a property that is not expressible in the infinitary logic $\mathscr{L}^\omega_{\infty\omega}$ (such as "there is an even number of elements"), then an infinite number of distinct counting quantifiers must occur in φ. Thus, it is natural to ask: is there a finite sequence $\boldsymbol{Q} = (Q_1, \ldots, Q_n)$ of simple unary quantifiers such that $\mathscr{L}^\omega_{\infty\omega}(\boldsymbol{Q})$ has the same expressive power as $\mathscr{L}^\omega_{\infty\omega}(\boldsymbol{C})$? This question is an instance of a more general problem, namely, given two collections $\boldsymbol{Q}$ and $\boldsymbol{Q}^*$ of generalized quantifiers on finite structures, how do the infinitary logics $\mathscr{L}^\omega_{\infty\omega}(\boldsymbol{Q})$ and $\mathscr{L}^\omega_{\infty\omega}(\boldsymbol{Q}^*)$ compare to each other in terms of expressive power? We use the pebble games for the logics $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, $k \geqslant 1$, to answer such questions.

Our main technical result is that there are natural polynomial-time properties on finite structures that are expressible by sentences of $\mathscr{L}^2_{\infty\omega}(\boldsymbol{C})$, but are not expressible by any sentence of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, where $\boldsymbol{Q}$ is a *finite* sequence of arbitrary simple unary generalized quantifiers and k is a positive integer. In particular, we establish that on the class of finite graphs no sentence of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ can express the query "do two given vertices have the same degree?". We also show that on the class of finite equivalence relations no sentence of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ can express the query "is there an even number of equivalence classes?". The proofs require the construction of structures such that, on the one hand, they satisfy the same sentences of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, but, on the other, they disagree on these queries. The required structures are constructed by means of a method we call the *$(k, \boldsymbol{Q})$-coloring method* combined with sophisticated tools from *Ramsey Theory*, such as *van der Waerden's Theorem* and *Folkman's Theorem*. Up to now constructions for lower-bound results in finite model theory have been either direct constructions, as in [16, 14, 9], or probabilistic constructions, as in [5]. Our approach of utilizing combinatorial principles in building structures is entirely different from the previously used techniques and opens the possibility of obtaining novel lower-bound results for expressibility in finite model theory.

All the individual extensions of first-order logic discussed above have been studied on finite structures in their own right, but no general theory of extended logics in finite model theory has been developed so far. It is natural to ask whether or not the framework of abstract model theory developed by logicians can provide such a general theory. As it turns out, finite model theory does not fall under the scope of the current framework of abstract model theory. This is so, because, although the concept of an extension of first-order logic in abstract model theory permits great flexibility in the syntax and the semantics, the framework is rather rigid when it comes to the part of the structures considered. More specifically, it is the case that always both finite and infinite structures are considered. In fact, the infinite structures play an indispensable role in many theorems of abstract model theory. This is manifested with Lindström's [40] theorem, which does not yield a characterization of first-order logic on finite structures, because the compactness theorem of first-order logic fails when only finite structures are considered.

In this paper we expand the framework of abstract model theory in a way that allows for a treatment of finite model theory. A multitude of lines of research emerges in this expanded framework, each with its own technical problems. Are there any interesting results in abstract model theory that carry over to finite structures? What can be said about the expressive power and the model-theoretic properties of first-order logic augmented with generalized quantifiers on finite structures? Is there a Lindström-type characterization of first-order logic or of fixpoint logic on finite structures? The latter question is of particular interest, because such a characterization of fixpoint logic may provide a deeper explanation for its eminence and robustness on finite structures.

A few results of abstract model theory still hold for abstract logics on an arbitrary class $\mathscr{K}$ of structures. This is, for example, the case with the well-known result that if the *Craig Interpolation Theorem* holds for an abstract logic L, then the *Beth Definability Theorem* also holds for L. In general, however, the results of abstract model theory do not necessarily carry over to abstract logics on an arbitrary class $\mathscr{K}$ of structures. This is particularly true when we consider the class $\mathscr{F}$ of all finite structures. It is evident that Lindström's characterizations of first-order logic on the class $\mathscr{S}$ of all structures do not hold on $\mathscr{F}$. For an example of a different flavor, consider the following: it is known that if *Robinson Consistency Theorem* holds for an abstract logic L on $\mathscr{S}$, then the Craig Interpolation Theorem holds for L ([42, 1.4]). This implication, however, does not hold on the class $\mathscr{F}$ of finite structures, since Robinson's theorem is trivially true for first-order logic on $\mathscr{F}$, while the Craig Interpolation Theorem fails.

In view of the above, it is natural to ask: is there a theory of abstract logics on classes $\mathscr{K}$ of structures other than the class $\mathscr{S}$ of all structures? In particular, is there a theory of abstract logics on the class $\mathscr{F}$ of all finite structures? Are there any model-theoretic properties that characterize first-order logic or one of its extensions on $\mathscr{F}$?

We feel that these are natural questions that merit further investigation. Our aim in this paper is to give some evidence that it is indeed both possible and sensible to develop a theory of abstract logics on finite structures. Actually, we believe that interesting results in this vein can be discovered by taking advantage of the finiteness of the structures and by viewing the absence of infinite structures as a feature, instead of an impediment. To illustrate this point, we present a Lindström-type characterization of the infinitary logics $\mathscr{L}^k_{\infty\omega}$ and $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, $k \geqslant 1$, on the class $\mathscr{F}$ of finite structures.

In order to make this paper as accessible as possible by newcomers to the area of generalized quantifiers and abstract model theory, we have included detailed definitions of the basic concepts and background material from this area.

2. Generalized quantifiers

2.1. *Background and definitions*

Historically, the research on extensions of first-order logic on infinite structures has its origins in the work of Mostowski [44] on *cardinality quantifiers*, such as "there is an infinite number of elements". Later on, Lindström [39] introduced *generalized quantifiers* and initiated the development of abstract model theory. Since that time, researchers in mathematical logic have investigated in depth the model theory of first-order logic augmented with specific generalized quantifiers, such as the quantifier "there exist uncountably many" (cf. [32]), and have analyzed the expressive power of first-order logic with various generalized quantifiers on infinite structures (cf. [8]). In contrast, generalized quantifiers on finite structures have gotten much less attention, in spite of an early pioneering paper by Hájek [24]. This situation, however, is changing rapidly. Indeed, recently the study of generalized quantifiers in the context of finite models has found applications in linguistics ([47, 31, 50, 51]) and computer science ([9, 30, 25]).

One of our main goals in this paper is to initiate a systematic investigation of generalized quantifiers on finite structures. To this effect, we are interested in both the expressive power and the model-theoretic properties of logics with generalized quantifiers on finite structures.

In this section we introduce formally generalized quantifiers and combine them with the infinitary logic $\mathscr{L}^\omega_{\infty\omega}$ with a finite number of variables.

We use $\mathscr{K}$ to denote an arbitrary class of structures. If σ is a vocabulary, we use $\mathscr{K}[\sigma]$ to denote the class of structures over σ that are in reducts of structures in $\mathscr{K}$. Some special cases will be used frequently: $\mathscr{S}$ denotes the class of all structures; $\mathscr{F}$ denotes the class of all *finite* structures. Let $\prec$ be a fixed binary predicate. We use $\mathscr{F}_\prec$ to denote the class of finite structures, one of whose relations is $\prec$ and in which $\prec$ is interpreted as a total ordering of the universe. Structures over the empty vocabulary $\emptyset$ are denoted by (A), where A is the universe of the structure. Thus $\mathscr{K}[\emptyset]$ consists of structures of the form (A).

Definition 2.1. A *simple unary generalized quantifier* is a class Q of structures over the vocabulary consisting of a unary relation symbol P such that Q is *closed under isomorphisms*, i.e., if $\boldsymbol{A} = (A, P^A)$ is a structure in Q and $\boldsymbol{B} = (B, P^B)$ is a structure that is isomorphic to $\boldsymbol{A}$, then $\boldsymbol{B}$ is also in Q. Let $\mathscr{K}$ be a class structures. A *simple unary generalized quantifier on* $\mathscr{K}$ is a simple unary generalized quantifier Q such that $Q[\emptyset] \subseteq \mathscr{K}[\emptyset]$.

Notice that the requirement on a simple unary generalized quantifier to be closed under isomorphisms is equivalent to a cardinality condition, namely, a simple unary generalized quantifier Q is a class of structures of the form $\boldsymbol{A} = (A, X)$ with $X \subseteq A$ and such that if $\boldsymbol{A} = (A, X)$ is in Q, $\boldsymbol{B} = (B, Y)$ is a structure with $Y \subseteq B$, $|X| = |Y|$, and $|A - X| = |B - Y|$, then $\boldsymbol{B}$ is also in Q. Moreover, on the class $\mathscr{F}$ of all finite structures this condition amounts to requiring that if $\boldsymbol{A} = (A, X)$ is in Q and $\boldsymbol{B} = (B, Y)$ is a structure with $Y \subseteq B$, $|A| = |B|$ and $|X| = |Y|$, then $\boldsymbol{B}$ is also in Q.

As mentioned above, simple unary generalized quantifiers on the class $\mathscr{S}$ of all structures were introduced by Mostowski [44]. In this framework, the *existential* quantifier on $\mathscr{S}$ is the class of all structures $\boldsymbol{A} = (A, X)$ with X a nonempty subset of A, while the *universal quantifier* consists of all structures of the form $\boldsymbol{A} = (A, A)$. Other canonical examples of simple unary generalized quantifiers on $\mathscr{S}$ are provided by *cardinality* quantifiers, such as the quantifier "there are at least $\aleph_\alpha$ elements". More formally, this quantifier is the class Q_α of all structures $\boldsymbol{A} = (A, X)$ with $X \subseteq A$ and $|X| \geqslant \aleph_\alpha$. If $\mathscr{K}$ is a class of structures, then the *Chang* quantifier over $\mathscr{K}$ is the class of all structures $\boldsymbol{A} = (A, X)$ such that $(A) \in \mathscr{K}[\emptyset]$ and X is a subset of A with $|X| = |A|$. In all the preceding examples the quantifiers share an important *monotonicity* property, which now we turn into a definition.

Definition 2.2. Let Q be a simple unary generalized quantifier. We say that Q is a *monotone* quantifier if for every structure $\boldsymbol{A} = (A, X)$ in Q and every subset Y of A such that $X \subseteq Y$, we have that the structure (A, Y) is also in Q.

The quantifier "there are exactly $\aleph_\alpha$ elements" provides a standard example of a simple unary generalized quantifiers on $\mathscr{S}$ that is not monotone. This quantifier is the class $Q_{=\alpha}$ of all structures $\boldsymbol{A} = (A, X)$ with $X \subseteq A$ and $|X| = \aleph_\alpha$. One advantage of monotone quantifiers over nonmonotone ones is that when they are added to a logic like $\mathscr{L}^{\omega}_{\infty\omega}$, then formulas, that are *positive* in a relation symbol S are also *monotone* in S, that is to say, if S has only positive occurrences in a formula, then the formula is preserved when new tuples are added in S.

There are plenty of natural examples of simple unary generalized quantifiers on any class of structures. Main emphasis in this paper is, however, on generalized quantifiers on the class $\mathscr{F}$ of all finite structures.

2.2. Simple unary generalized quantifiers on finite structures

We next consider simple unary generalized quantifiers on the class $\mathscr{F}$ of all finite structures. Notice that on finite structures all cardinality quantifiers Q_α reduce to the trivial empty quantifier, while the Chang quantifier on $\mathscr{F}$ coincides with the universal quantifier. On finite structures the analog of the cardinality quantifiers are the *counting* quantifiers $(\exists i x)$, $i \geqslant 1$, consisting of all structures $\boldsymbol{A} = (A, X)$, where A is a finite set and X is a subset of A with at least i elements. The counting quantifiers are, of course, expressible using the existential quantifier and first-order logic, but, as explained earlier, they become quite interesting when we consider them in the context of logics with a fixed number of variables.

Numerous natural examples of simple unary generalized quantifiers on $\mathscr{F}$ arise from properties that are not first-order definable on finite structures, such as "there is an even number of elements", "there are at least $\log(n)$ many elements" (where n is the cardinality of the structure), or "the number of elements satisfying ... divides the cardinality of the structure" (the *divisibility* quantifier). In particular, the quantifier "there is an even number of elements" can be viewed as the class

$$Q_{\text{even}} = \{(A, X)\colon A \text{ is a finite set, } |X| \subseteq A, \text{ and } |X| \text{ is even}\}.$$

Notice that the counting quantifiers and the quantifier

$$Q_{\text{half}} = \{(A, X)\colon A \text{ is a finite set, } |X| \subseteq A, \text{ and } |X| \geqslant |A|/2\}.$$

are monotone, while the quantifier Q_{even} and the divisibility quantifier are not.

A useful insight to the structure of simple unary generalized quantifiers on $\mathscr{F}$ can be obtained by associating with each such quantifier a function that gives the cardinalities of sets occurring in the quantifier.

Definition 2.3. Let Q be a simple unary generalized quantifier on the class $\mathscr{F}$ of all finite structures. The *defining function f_Q of the quantifier Q* is the function with domain the set of all positive integers n and values

$$f_Q(n) = \{m\colon 0 \leqslant m \leqslant n \text{ and there is a structure } (A, X) \in Q \text{ such that } |X| = m\}.$$

The requirement that simple unary generalized quantifiers be closed under isomorphisms implies that the defining function of such a quantifier characterizes the quantifier, in the sense that for two simple unary generalized quantifiers Q_1 and Q_2 we have that

$$Q_1 = Q_2 \text{ if and only if } f_{Q_1} = f_{Q_2}.$$

Notice that if Q is a monotone simple unary generalized quantifier on $\mathscr{F}$, then for each $n \geqslant 1$ the value $f_Q(n)$ of the defining function is the interval

$$[r_Q(n), n] = \{m\colon m \text{ is a nonnegative integer and } r_Q(n) \leqslant m \leqslant n\},$$

where $r_Q(n) = \min\{m: m \in f_Q(n)\}$. In this case, Q is the quantifier "there are at least $r_Q(n)$ elements", i.e., it is the class

$$Q = \{(A, X): A \text{ is a finite set, } X \subseteq A, \text{ and } |X| \geqslant r_Q(|A|)\}.$$

We conclude that the mapping $n \mapsto r_Q(n)$ is a function from the positive integers to the nonnegative integers that describes completely the monotone quantifier Q. Conversely, every function $r(n)$ from the positive integers to the nonnegative integers gives rise to a monotone quantifier, namely the quantifier "there are at least $r(n)$ elements". It follows that there is a one-to-one and onto correspondence between monotone simple unary generalized quantifiers Q on $\mathscr{F}$ and arbitrary functions $r(n)$ from the positive integers to the nonnegative integers.

It turns out that, by reflecting on the properties of the function $r_Q(n)$, we can obtain a simple classification of all monotone quantifiers Q on $\mathscr{F}$.

Definition 2.4. Let Q be a monotone simple unary generalized quantifier on the class $\mathscr{F}$ of all finite structures and for each $n \geqslant 1$ let $r_Q(n) = \min\{m: m \in f_Q(n)\}$, where $f_Q(n)$ is the defining function of Q. If $\{m: m \in f_Q(n)\} = \emptyset$, then we let $r_Q(n) = n + 1$.

- We say that Q is an *eventually counting* quantifier if there is a positive integer N and a nonnegative integer r such that one of the following two statements holds:
  1. $r_Q(n) = r$, for all positive integers $n \geqslant N$.
  2. $r_Q(n) = n - r$, for all positive integers $n \geqslant N$.
- We say that Q is an *eventually bounded* quantifier if there is a positive integer N, and two finite sets $S_1 = \{r_1, \ldots, r_l\}$ and $S_2 = \{s_1, \ldots, s_m\}$ of nonnegative integers such that the following hold:
  1. $S_1 \cup S_2$ is nonempty and if one of the sets S_1 and S_2 is empty, then either the other set has at least two elements or $f(n) = n + 1$ for infinitely many n.
  2. For every $n \geqslant N$ there is a j such that either $1 \leqslant j \leqslant l$ and $r_Q(n) = r_j$, or $1 \leqslant j \leqslant m$ and $r_Q(n) = n - s_j$, or $f(n) = n + 1$.
  3. The function $r_Q(n)$ takes each one of the values r_j, $1 \leqslant j \leqslant l$, and $n - s_j$, $1 \leqslant j \leqslant s$, infinitely often.
- We say that Q is an *unbounded* quantifier if there is an infinite increasing sequence $n_1 < n_2 < \cdots < n_i < n_{i+1} < \cdots$ of positive integers such that

  $$r_Q(n_1) < r_Q(n_2) < \cdots < r_Q(n_i) < r_Q(n_{i+1}) < \cdots$$

  and

  $$n_1 - r_Q(n_1) < n_2 - r_Q(n_2) < \cdots < n_i - r_Q(n_i) < n_{i+1} - r_Q(n_{i+1}) < \cdots$$

The counting quantifiers $(\exists i x)$, $i \geqslant 1$, and their *dual* quantifiers "there are at least $n - i$ elements", $i \geqslant 1$, are the main examples of eventually counting quantifiers. The quantifiers "there are at least $\log(n)$ elements", "there are at least $\sqrt{n}$ elements", and "there are at least $n/2$ elements" are all examples of unbounded quantifiers. Finally, the following are examples of eventually bounded quantifiers:

- $Q = \{(A, X)\colon (|A| \text{ is even and } |X| \geqslant 3) \text{ or } (|A| \text{ is odd and } |X| \geqslant 5)\}$;
- $Q = \{(A, X)\colon (|A| \text{ is even and } |X| \geqslant 3) \text{ or } (|A| \text{ is odd and } |X| \geqslant |A| - 5)\}$;
- $Q = \{(A, X)\colon |A| = i \bmod 3 \text{ and } |X| \geqslant |A| - i,\ i = 0, 1, 2\}$.

The next result yields a classification of all monotone simple unary generalized quantifiers on finite structures. The proof follows easily from the definitions.

Proposition 2.5. *Let Q be a monotone simple unary generalized quantifier on the class $\mathscr{F}$ of all finite structures. Then exactly one of the following three statements holds:*
1. *Q is an eventually counting quantifier;*
2. *Q is an eventually bounded quantifier;*
3. *Q is an unbounded quantifier.*

The above classification will be used later on in order to establish a structural characterization of the eventually counting quantifiers. Moreover, we will show that quantifiers definable in $\mathscr{L}^{\omega}_{\infty\omega}$ from eventually bounded quantifiers are themselves eventually bounded or counting.

We can think of generalized quantifiers on $\mathscr{F}$ as *queries*. Thus a quantifier Q on $\mathscr{F}$ corresponds to the query "is a given finite structure in $\boldsymbol{Q}$?". If we think of quantifiers on $\mathscr{F}$ as queries, it is clear what it means for a quantifier to be PTIME computable. In any standard coding of finite structures the length of the code of a structure is polynomial in the size of the structure. A simple unary generalized quantifier Q can be defined to be PTIME if there is an algorithm which decides whether a given structure $\boldsymbol{A}$ is in Q or not, and stops in time which is polynomial in the code of $\boldsymbol{A}$.

2.3. *Infinitary logics with generalized quantifiers*

We now define the syntax and the semantics of the logics that are obtained by combining simple unary generalized quantifiers with the infinitary logics $\mathscr{L}^{k}_{\infty\omega}$, $k \geqslant 1$.

Definition 2.6. Let $\boldsymbol{Q} = \{Q_i\colon i \in I\}$ be a family of simple unary generalized quantifiers, and let k be a positive integer. The *infinitary logic $\mathscr{L}^{k}_{\infty\omega}(\boldsymbol{Q})$ with k variables and the generalized quantifiers $\boldsymbol{Q}$* has the following syntax (for any vocabulary σ):
- The variables of $\mathscr{L}^{k}_{\infty\omega}(\boldsymbol{Q})$ are $v_1, \ldots, v_k$;
- $\mathscr{L}^{k}_{\infty\omega}(\boldsymbol{Q})$ contains all first-order formulas over σ with variables among $v_1, \ldots, v_k$;
- if φ is a formula of $\mathscr{L}^{k}_{\infty\omega}(\boldsymbol{Q})$, then so is $\neg\varphi$;
- if Ψ is a set of formulas of $\mathscr{L}^{k}_{\infty\omega}(\boldsymbol{Q})$, then $\bigvee\Psi$ and $\bigwedge\Psi$ are also formulas of $\mathscr{L}^{k}_{\infty\omega}(\boldsymbol{Q})$;
- if φ is a formula of $\mathscr{L}^{k}_{\infty\omega}(\boldsymbol{Q})$, then each of the expressions $\exists v_j\varphi, \forall v_j\varphi, Q_i v_j\varphi$ is also a formula of $\mathscr{L}^{k}_{\infty\omega}(\boldsymbol{Q})$ for every j such that $1 \leqslant j \leqslant k$ and for every $i \in I$.

Notice that although there are only k distinct variables, a sentence of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ may have infinitely many occurrences of a variable. The concepts of a *free* and *bound* variable in a formula of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ are defined in the same way as in first-order logic with the additional stipulation that the variable v_j is bound in the formula $Q_i v_j \varphi$. A sentence of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ is a formula of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ with no free variables.

The *semantics* of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ is defined by induction on the construction of the formulas. More specifically, $\bigvee\Psi$ is interpreted as a disjunction over all formulas in Ψ and $\bigwedge\Psi$ is interpreted as a conjunction. Finally, if $\boldsymbol{A}$ is a structure having A as its universe and $\varphi(v_j, \boldsymbol{y})$ is a formula of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ with free variables among the variables v_j and the variables in the sequence $\boldsymbol{y}$, and $\boldsymbol{d}$ is a sequence of elements from the universe of $\boldsymbol{A}$, then

$$\boldsymbol{A}, \boldsymbol{d} \models Q_i v_j \varphi(v_j, \boldsymbol{y})$$

if and only if the structure

$$(A, \{a: \boldsymbol{A}, a, \boldsymbol{d} \models \varphi(v_j, \boldsymbol{y})\})$$

is in the quantifier Q_i.

We write $\mathscr{L}^\omega_{\infty\omega}(\boldsymbol{Q})$ to denote the union $\bigcup_{k=1}^{\infty} \mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ of the infinitary logics with a finite number of variables and the generalized quantifiers $\boldsymbol{Q}$. If $\boldsymbol{Q}$ is a finite sequence $(Q_1, \ldots, Q_n)$ of simple unary generalized quantifiers, then we write $\mathscr{L}^k_{\infty\omega}(Q_1, \ldots, Q_n)$, $k \geqslant 1$, and $\mathscr{L}^\omega_{\infty\omega}(Q_1, \ldots, Q_n)$ in place of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ and $\mathscr{L}^\omega_{\infty\omega}(\boldsymbol{Q})$, respectively.

If the definition of the syntax of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, $k \geqslant 1$, is modified by requiring that the disjunctions and conjunctions are always applied to finite set of formulas, then we obtain the logic $\mathscr{L}^k_{\omega\omega}(\boldsymbol{Q})$, which is the fragment of first-order logic with k variables and the generalized quantifiers $\boldsymbol{Q}$. The union of these logics gives us $\mathscr{L}_{\omega\omega}(\boldsymbol{Q})$, first-order logic augmented with the generalized quantifiers $\boldsymbol{Q}$.

It should be pointed out that the expressive power of $\mathscr{L}^\omega_{\infty\omega}(\boldsymbol{Q})$ transcends properly the expressive power of both $\mathscr{L}_{\omega\omega}(\boldsymbol{Q})$ and $\mathscr{L}^\omega_{\infty\omega}$. This is, for example, the case when $\boldsymbol{Q}$ is the family $\boldsymbol{C}$ of all counting quantifiers $(\exists i x)$, $i \geqslant 1$. Moreover, the property "there is an even number of articulation points" is easily expressible in $\mathscr{L}^3_{\infty\omega}(Q_{\text{even}})$ on finite graphs, although one can prove that it is not expressible neither in $\mathscr{L}^\omega_{\infty\omega}$, nor in $\mathscr{L}_{\omega\omega}(Q_{\text{even}})$. A similar fact holds for the property "there is a connected component that has at least half the nodes" and the quantifier "for at least half the nodes".

The model-theoretic properties of the logic $\mathscr{L}_{\omega\omega}(\boldsymbol{Q})$ on the class of all structures have been investigated in depth for various quantifiers $\boldsymbol{Q}$ arising in mathematical practice (cf. [8]). As mentioned earlier, on finite structures the family of the counting quantifiers $\boldsymbol{C} = \{(\exists i x)\colon i \geqslant 1\}$ and the resulting infinitary logics $\mathscr{L}^k_{\infty\omega}(\boldsymbol{C})$, $k \geqslant 1$ have been studied systematically by Cai et al. [9] and by Immerman and Lander [30]. So far, the infinitary logics $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, $k \geqslant 1$, have not been explored neither on infinite nor on finite structures for other unary generalized quantifiers.

It is simple, but important, fact that on finite structures the infinitary logic $\mathscr{L}^\omega_{\infty\omega}(\boldsymbol{C})$ can subsume every logic $\mathscr{L}^\omega_{\infty\omega}(\boldsymbol{Q})$, where $\boldsymbol{Q}$ is an arbitrary family of (simple) unary generalized quantifiers. A more general result will be proved in Section 2.5.

Proposition 2.7. *Let $\boldsymbol{Q} = \{Q_i\colon i \in I\}$ be a family of simple unary generalized quantifiers on a class $\mathscr{K}$ of finite structures and let k be a positive integer. If $\psi(\boldsymbol{y})$ is a formula of the infinitary logic $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, then there is a formula $\psi^*(\boldsymbol{y})$ of the infinitary logic $\mathscr{L}^k_{\infty\omega}(\boldsymbol{C})$ with counting quantifiers and k variables such that $\psi^*(\boldsymbol{y})$ is equivalent to $\psi(\boldsymbol{y})$ on all structures in $\mathscr{K}$.*

Proof. The proof is by induction on the construction of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ formulas. The only interesting case is when $\psi(\boldsymbol{y})$ is of the form $Q_i x\varphi(x,\boldsymbol{y})$ for some quantifier Q_i in the family $\boldsymbol{Q}$. By induction hypothesis, assume that $\varphi(x,\boldsymbol{y})$ is equivalent to a formula $\varphi^*(x,\boldsymbol{y})$ of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{C})$. Let S be the set of all integers that are cardinalities of structures in $\mathscr{K}$ and let f_{Q_i} be the defining function of the quantifier Q_i. Then $\psi(\boldsymbol{y})$ is equivalent on $\mathscr{K}$ to the formula

$$\bigvee_{n\in S}\left((\exists!nx)(x=x)\wedge\left[\bigvee_{m\in f_{Q_i}(n)}(\exists!mx)\varphi^*(x,\boldsymbol{y})\right]\right)$$

of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{C})$, where $\exists!nx(\ldots)$ is an abbreviation for $(\exists nx)(\ldots)\wedge\neg(\exists n+1x)(\ldots)$. □

The main result of Cai et al. [9] asserts that there are polynomial-time properties of graphs that are not expressible by any formula of the infinitary logic $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$ on finite graphs. Combined with Proposition 2.7 yields immediately the following result.

Corollary 2.8. *Let $\boldsymbol{Q} = \{Q_i\colon i \in I\}$ be a family of simple unary generalized quantifiers on the class $\mathscr{K}$ of all finite graphs. Then there are polynomial-time properties that are not expressible by any formula of the infinitary logic $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$ on $\mathscr{K}$.*

Corollary 2.8 reveals that there is no hope of capturing all of PTIME by combining simple unary generalized quantifiers with the infinitary logic $\mathscr{L}^{\omega}_{\infty\omega}$. At the same time, it raises a number of interesting questions concerning the properties of the infinitary logics $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$ for families $\boldsymbol{Q}$ of simple unary generalized quantifiers.

Notice that if a formula φ of $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{C})$ defines a property that is not expressible in the infinitary logic $\mathscr{L}^{\omega}_{\infty\omega}$, then an infinite number of different counting quantifiers will occur in φ. This raises the question: is there a finite family $\boldsymbol{Q}$ of simple unary generalized quantifiers such that $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$ has the same expressive power as $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{C})$?

More generally, given two families $\boldsymbol{Q}$ and $\boldsymbol{Q}^*$ of simple unary generalized quantifiers on finite structures, we may ask, how do the infinitary logics $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$ and $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q}^*)$ compare to each other in terms of expressive power? In the next section we will develop tools for studying and answering these questions.

2.4. Fixpoint logic with monotone quantifiers

The monotonicity condition is an important restriction to the concept of a generalized quantifier. One immediate consequence of this condition is that we can use the concept of a positive occurrence of a predicate symbol to get a profusion of definable monotone operators and we can define fixpoint logic with monotone quantifiers. More specifically, if $\boldsymbol{Q} = \{Q_i: i \in I\}$ is a collection of monotone simple unary generalized quantifiers, we can define *fixpoint logic with the quantifiers* $\boldsymbol{Q}$ as the extension $FP(\boldsymbol{Q})$ of first-order logic obtained by augmenting the syntax of $\mathscr{L}_{\omega\omega}(\boldsymbol{Q})$ with least fixpoints of positive formulas. Below we shall give a more detailed definition.

So far, fixpoint logic with monotone quantifiers has been studied mainly on infinite structures (cf. [3]). We investigate the expressive power and the closure properties of $\mathrm{FP}(\boldsymbol{Q})$ on finite structures. It turns out, for example, that every formula of $\mathrm{FP}(\boldsymbol{Q})$ is equivalent to a formula of $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$. Moreover, it is easy to see that if the defining function f_{Q_i} of every quantifier Q_i in $\boldsymbol{Q}$ is computable in polynomial time, then every $\mathrm{FP}(\boldsymbol{Q})$ query is in polynomial time. As far as closure properties are concerned, we point out that $\mathrm{FP}(\boldsymbol{Q})$ queries are closed under complements on finite structures.

Definition 2.9. The *dual* of a simple unary quantifier Q is the simple unary quantifier

$$\breve{Q} = \{(A, X): X \subseteq A \text{ and } (A, A - X) \notin Q\}.$$

Note that $\breve{Q}$ is definable in $\mathscr{L}^1_{\omega\omega}(Q)$ and if Q is monotone, then so is $\breve{Q}$. If $\boldsymbol{Q} = \{Q_i: i \in I\}$ is a collection of monotone simple unary generalized quantifiers, then every formula in $\mathscr{L}_{\omega\omega}(\boldsymbol{Q})$ can be expressed in an equivalent form which has only atomic formulas, their negations, connectives $\wedge$, $\vee$ and quantifiers $\exists, \forall, Q_i, \breve{Q}_i$, where $i \in I$.

Definition 2.10. Suppose $\boldsymbol{Q} = \{Q_i: i \in I\}$ is a collection of monotone simple unary generalized quantifiers and $\varphi \in \mathscr{L}_{\omega\omega}(\boldsymbol{Q})$. Suppose φ is written so that it contains only atomic formulas, their negations, connectives $\wedge$, $\vee$ and quantifiers $\exists, \forall, Q_i, \breve{Q}_i$, where $i \in I$. We say that the occurrence of a predicate symbol in φ is *positive* if it is immediately preceded by an even number of negation symbols.

The following lemma is easy to prove.

Lemma 2.11. *Suppose S is an n-ary relation symbol not in the vocabulary σ and $\varphi(x_1, \ldots, x_n, S)$ is a formula of $\mathscr{L}_{\omega\omega}(\boldsymbol{Q})$ over the vocabulary $\sigma \cup \{S\}$ in which S has only positive occurrences. Let*

$$\varphi^0(x_1, \ldots, x_n) = \neg x_1 = x_1,$$

$$\varphi^{i+1}(x_1, \ldots, x_n) = \varphi(x_1, \ldots, x_n, S(t_1, \ldots, t_n)/\varphi^i(t_1, \ldots, t_n)),$$

$$\varphi^{\infty}(x_1, \ldots, x_n) = \bigvee \{\varphi^i(x_1, \ldots, x_n): i = 0, 1, 2, \ldots\}.$$

Suppose $\boldsymbol{A}$ is a finite σ-structure and

$$S = \{(a_1, \ldots, a_n): \boldsymbol{A}, a_1, \ldots, a_n \models \varphi^{\infty}(x_1, \ldots, x_n)\}.$$

Then S is the smallest fixed point of $\varphi(x_1, \ldots, x_n, S)$ on A i.e. the smallest n-ary relation S on A such that

$$(\forall a_1, \ldots, \forall a_n)(S(a_1, \ldots, a_n) \Leftrightarrow A, a_1, \ldots, a_n \models \varphi(x_1, \ldots, x_n, S)).$$

Here $\varphi(x_1, \ldots, x_n, S(t_1, \ldots, t_n)/\varphi^i(t_1, \ldots, t_n))$ refers to the formula which is obtained from $\varphi(x_1, \ldots, x_n, S(t_1, \ldots, t_n))$ by replacing $S(t_1, \ldots, t_n)$ everywhere by $\varphi^i(t_1, \ldots, t_n)$ for all sequences $t_1, \ldots, t_n$ of terms.

Definition 2.12. Suppose $\boldsymbol{Q} = \{Q_i\colon i \in I\}$ is a collection of monotone simple unary generalized quantifiers. We let FP($\boldsymbol{Q}$) be the smallest collection of formulas that contains $\mathscr{L}_{\omega\omega}(\boldsymbol{Q})$, least fixpoints φ^∞ of formulas in $\mathscr{L}_{\omega\omega}(\boldsymbol{Q})$, and is closed under finitary disjunctions and conjunctions, existential and universal quantification, and Q_i and $\breve{Q}_i$ quantification for $i \in I$.

Immerman [28] considered fixpoint logic with the counting quantifiers $\boldsymbol{C}$ on certain two-sorted finite structures, where one of the sorts is used to do arithmetic on the counting quantifiers. In the context discussed here, FP($\boldsymbol{C}$) has the same expressive power as fixpoint logic FP, while FP(Q_{half}) is strictly more expressive than FP. By imitating the usual proof that fixpoint logic on finite structures is contained in $\mathscr{L}^k_{\infty\omega}$ ([7]) and the proof that fixpoint queries are PTIME, one can show the following proposition.

Proposition 2.13. *Suppose $\boldsymbol{Q} = \{Q_i\colon i \in I\}$ is a collection of monotone simple unary PTIME generalized quantifiers. Then on finite structures:*
1. *Every formula of* FP($\boldsymbol{Q}$) *is expressible in $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$.*
2. *Every sentence in* FP($\boldsymbol{Q}$) *defines a PTIME query.*

Leivant [38] defines the concept of a *monotone language*. By this he means a language L in which the concept of "positive occurrence" is defined and which satisfies the conditions:
1. L is closed under first-order operations.
2. For every formula φ and relation symbol R positive in φ, the formula

   $$\forall z(P(z) \rightarrow Q(z)) \rightarrow (\varphi[P/R] \rightarrow \varphi[Q/R])$$

   is valid.

It is clear that if $\boldsymbol{Q} = \{Q_i\colon i \in I\}$ is a collection of monotone simple unary generalized quantifiers, then FP($\boldsymbol{Q}$) and $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ are monotone languages in the sense of Leivant. The following result follows from [38, Lemmas 3, 4 and Theorem IV].

Proposition 2.14. *Suppose $\boldsymbol{Q} = \{Q_i\colon i \in I\}$ is a collection of monotone simple unary generalized quantifiers. Then* FP($\boldsymbol{Q}$) *is closed under negation, i.e. the negation of a formula of* FP($\boldsymbol{Q}$) *is always expressible in* FP($\boldsymbol{Q}$).

2.5. *Lindström quantifiers*

The concept of a simple unary generalized quantifier was extended to quantifiers of arbitrary arity on the class $\mathscr{S}$ of all structures by Lindström [39].

Definition 2.15. Let n be a positive integer. An *n-ary generalized quantifier* is a class Q of structures over the vocabulary consisting of an n-ary relation symbol P such that Q is *closed under isomorphisms*, i.e., if $\boldsymbol{A} = (A, P^A)$ is a structure in Q and $\boldsymbol{B} = (B, P^B)$ is a structure that is isomorphic to $\boldsymbol{A}$, then $\boldsymbol{B}$ is also in Q. Let $\mathscr{K}$ be a class of structures and n a positive integer. An *n-ary generalized quantifier on* $\mathscr{K}$ is an n-ary generalized quantifier Q such that $Q[\emptyset] \subseteq \mathscr{K}[\emptyset]$.

Every class of equivalence relations that is closed under isomorphisms gives rise to a binary generalized quantifiers on $\mathscr{S}$. In particular, such a quantifier is provided by the class of all structures $\boldsymbol{A} = (A, E)$ with the property that E is an equivalence relation on A having infinitely many equivalence classes. For an example of a different nature, consider the *well-ordering quantifier* $Q^{\#}$ on the class $\mathscr{S}$ of all structures: it consists of all structures $\boldsymbol{A} = (A, <^A)$ such that the binary relation $<^A$ is a well-ordering of A. This quantifier is not expressible in first-order logic. Observe, however, that the restriction of the *well-ordering* quantifier $Q^{\#}$ to the class $\mathscr{F}$ of all finite structures is first-order definable, since on finite structures well-orderings coincide with total-orderings.

In general, every collection of finite graphs that is closed under isomorphisms gives rise to a binary generalized quantifier on the class $\mathscr{F}$ of all finite structures. For example, the *connectivity quantifier* consists of all finite connected graphs $G = (V, E)$.

With only notational modifications in Definition 2.6, we can define the syntax and semantics of the logics $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, $k \geqslant 1$, for families $\boldsymbol{Q} = \{Q_i\colon i \in I\}$ of generalized quantifiers in which the arity of each quantifier Q_i is at most k.

Notice that n-ary generalized quantifiers always apply to a single formula. Lindström [39] introduced more complex quantifiers that can apply to a pair of formulas or even to a finite sequence of formulas.

Definition 2.16. Let $(n_1, \ldots, n_l)$ be a sequence of positive integers. A *Lindström quantifier of type* $(n_1, \ldots, n_l)$ is a class Q of structures over the vocabulary consisting of relation symbols $P_1, \ldots, P_l$ such that P_i is n_i-ary for $1 \leqslant i \leqslant l$ and Q is closed under isomorphisms. Let $\mathscr{K}$ be a class of structures and let $(n_1, \ldots, n_l)$ be a sequence of positive integers. A *Lindström quantifier of type* $(n_1, \ldots, n_l)$ *on* $\mathscr{K}$ is a Lindström quantifier Q of type $(n_1, \ldots, n_l)$ such that $Q[\emptyset] \subseteq \mathscr{K}[\emptyset]$.

Notice that n-ary generalized quantifiers are Lindström quantifiers of type (n). In the literature, n-ary generalized quantifiers are also known as *simple* Lindström quantifiers. One of the best-known examples of nonsimple quantifiers is the *equicardinality or Härtig* quantifier I. This is the Lindström quantifier of type $(1, 1)$ that consists of all structures $\boldsymbol{A} = (A, X, Y)$ such that $X \subseteq A$, $Y \subseteq A$, and $|X| = |Y|$. There has been an extensive study of the model-theoretic properties of the Härtig quantifier I on the class $\mathscr{S}$ of all structures (cf. [26]). Moreover, Chandra and Harel [11] showed that on finite structures the Härtig quantifier is not expressible in fixpoint logic. The *similarity quantifier* S_n, $n \geqslant 2$, is the higher arity analog of the Härtig quantifier: it is of type (n, n) and consists of all structures $\boldsymbol{A} = (A, R, S)$ such that $R \subseteq A^n$, $S \subseteq A^n$, and the structure (A, R) is isomorphic to the structure (A, S). Another well-studied Lindström quantifier of type $(1, 1)$ is the quantifier MORE that consists of all structures $\boldsymbol{A} = (A, X, Y)$ such that $X \subseteq A$, $Y \subseteq A$, and $|X| > |Y|$. This is also known as the *Rescher* quantifier.

We can now define the syntax and the semantics of the infinitary logics $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, $k \geqslant 1$, for families $\boldsymbol{Q} = \{Q_i: i \in I\}$ of arbitrary Lindström quantifiers, provided that the type $(n_1, \dots, n_l)$ of every quantifier Q_i in $\boldsymbol{Q}$ satisfies the inequality $\max\{n_1, \dots, n_l\} \leqslant k$.

At the level of syntax, we add the following construct: Let Q_i be a quantifier in $\boldsymbol{Q}$ of type $(n_1, \dots, n_l)$, let $\boldsymbol{x}_j$, $1 \leqslant j \leqslant l$, be n_j-tuples of variables of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, let $\boldsymbol{y}$ be a sequence of variables of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, and let $\varphi_i(\boldsymbol{x}_i, \boldsymbol{y})$, $1 \leqslant j \leqslant l$, be formulas of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ with free variables among the variables in the tuple $\boldsymbol{x}_j$ and the tuple $\boldsymbol{y}$. We assume that the tuples $\boldsymbol{x}_j$ and $\boldsymbol{y}$ are all disjoint. Then the expression

$$Q_i(\boldsymbol{x}_1, \dots, \boldsymbol{x}_l)(\varphi_1(\boldsymbol{x}_1, \boldsymbol{y}), \dots, \varphi_l(\boldsymbol{x}_l, \boldsymbol{y}))$$

is also a formula of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$. A structure $\boldsymbol{A}$ with universe A and a tuple $\boldsymbol{d}$ of elements from the universe of $\boldsymbol{A}$ satisfy the above sentence if and only if the structure

$$(A, \{\boldsymbol{a}_1: \boldsymbol{A}, \boldsymbol{a}_1, \boldsymbol{d} \vDash \varphi_1(\boldsymbol{x}_1, \boldsymbol{y})\}, \dots, \{\boldsymbol{a}_l: \boldsymbol{A}, \boldsymbol{a}_l, \boldsymbol{d} \vDash \varphi_l(\boldsymbol{x}_1, \boldsymbol{y})\})$$

is in the quantifier Q_i.

In particular, for the Härtig quantifier I we have that

$$\boldsymbol{A}, \boldsymbol{d} \vDash I(x_1, x_2)(\varphi(x_1, \boldsymbol{y}), \psi(x_2, \boldsymbol{y}))$$

if and only if

$$|\{a \in A: \boldsymbol{A}, a, \boldsymbol{d} \vDash \varphi(x_1, \boldsymbol{y})\}| = |\{b \in A: \boldsymbol{A}, b, \boldsymbol{d} \vDash \psi(x_2, \boldsymbol{y})\}|.$$

Note that the Härtig quantifier is readily definable from MORE. It was proved in [50] that MORE is not definable in $\mathscr{L}_{\omega\omega}(I)$. The exact relationship between these two quantifiers will be investigated in Section 5.

Definition 2.17. A Lindström quantifier Q of type $(n_1, \dots, n_l)$ on $\mathscr{F}$ is *numerical* if $(A, R_1, \dots, R_{n_l}) \in Q$ and $|R_1| = |R'_1|, \dots, |R_{n_l}| = |R'_{n_l}|$ imply $(A, R'_1, \dots, R'_{n_l}) \in Q$.

Intuitively, a quantifier on $\mathscr{F}$ is numerical if it only refers to cardinalities of relations. Simple unary quantifiers on $\mathscr{F}$, the Härtig quantifier I and MORE are good examples of numerical generalized quantifiers. We can easily think of other natural ones, like:

$$Q = \{(A, R): R \subseteq A^n, |R| \geqslant |A|^n/2\},$$

$$Q = \{(A, R): R \subseteq A^n, |R| \text{ even}\},$$

$$Q = \{(A, R, S): R, S \subseteq A^n, |R| = |S|\}.$$

Proposition 2.18. *Let $\boldsymbol{Q} = \{Q_i: i \in I\}$ be a family of numerical Lindström quantifiers on a class $\mathscr{K}$ of finite structures and let k be a positive integer. If $\psi(\boldsymbol{y})$ is a formula of the infinitary logic $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, then there is a formula $\psi^*(\boldsymbol{y})$ of the infinitary logic $\mathscr{L}^k_{\infty\omega}(\boldsymbol{C})$ with counting quantifiers and k variables such that $\psi^*(\boldsymbol{y})$ is equivalent to $\psi(\boldsymbol{y})$ on structures in $\mathscr{K}$.*

Proof. We use $[n]$ to denote the set $\{1, \ldots, n\}$ and $[n, m]$ to denote the set $\{n, n+1, \ldots, m\}$. Let Q_i be a numerical generalized quantifier of type $(n_1, \ldots, n_l)$. We prove that the sentence $Q_i(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_l)(P_1(\boldsymbol{x}_1), \ldots, P_l(\boldsymbol{x}_l))$ is definable in $\mathscr{L}^k_{\infty\omega}(\boldsymbol{C})$. The rest of the proof goes as in the proof of Proposition 2.7.

Let S_n be the set of tuples $s = (s_1, \ldots, s_q)$ of positive integers such that $s_1 + \cdots + s_q = n$. We denote the length q of s by $lh(s)$. Let

$$T^1_n = \{(s, f): s = (s_1), s_1 \in [0, n], f = \{(1, 0)\}\},$$

$$T^{i+1}_n = \{(s, f): s \in S_n, f: [lh(s)] \to T^i_n \text{ is one to one}\}.$$

Suppose $t = (s, f) \in T^{n_j - m}_n$, where $0 \leqslant m < n_j$. Let

$$\varphi^{j,0}_n = P_j(x_1, \ldots, x_{n_j}),$$

$$\varphi^{j,t}_n = \bigwedge_{i=1}^{lh(s)} \exists!\, s_i x_{m+1}\, \varphi^{j, f(i)}_n(x_1, \ldots, x_{m+1}).$$

Suppose $t = (s, f) \in T^{n_j}_n$. A structure $([n], R_j)$, $R_j \subseteq [n]^{n_j}$, satisfies $\varphi^{j,t}_n$ if and only if $|R_j| = c(t)$, where the number $c(t)$ is defined as follows:

$$c(0) = 1, \qquad c(t) = \sum_{i=1}^{lh(s)} s_i \cdot c(f(i)).$$

Now, a structure $([n], R_1, \ldots, R_l)$ is in Q_i if and only if it satisfies the sentence:

$$\bigvee \{\exists!\, nx(x = x) \wedge \varphi^{1,t_1}_n \wedge \cdots \wedge \varphi^{l,t_l}_n : t_i \in T^{n_i}_n \text{ for } i = 1, \ldots, l \text{ and } ([n], c(t_1), \ldots, c(t_l)) \in Q_i\}. \qquad \square$$

If we combine the main result of Cai et al. [9] and Proposition 2.18, we get the following corollary.

Corollary 2.19. *Let $\boldsymbol{Q} = \{Q_i: i \in I\}$ be a family of numerical Lindström quantifiers on the class $\mathscr{K}$ of all finite graphs. Then there are polynomial-time properties that are not expressible by any formula of the infinitary logic $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ on $\mathscr{K}$.*

Let us call a Lindström quantifier of type $(1, 1, \ldots, 1)$ *unary*. Not all unary quantifiers are numerical, for example the class of structures (A, X, Y), where $X, Y \subseteq A$ and $|X \cap Y|$ is even. However, Proposition 2.18 and Corollary 2.19 hold for all unary Lindström quantifiers.

Proposition 2.20. *Let $\mathbf{Q} = \{Q_i\colon i \in I\}$ be a family of unary Lindström quantifiers on a class $\mathscr{K}$ of finite structures and let k be a positive integer. If $\psi(\mathbf{y})$ is a formula of the infinitary logic $\mathscr{L}^k_{\infty\omega}(\mathbf{Q})$, then there is a formula $\psi^*(\mathbf{y})$ of the infinitary logic $\mathscr{L}^k_{\infty\omega}(\mathbf{C})$ with counting quantifiers and k variables such that $\psi^*(\mathbf{y})$ is equivalent to $\psi(\mathbf{y})$ on structures in $\mathscr{K}$. Moreover, if $\mathscr{K}$ is the class of finite graphs, then there are polynomial-time properties that are not expressible by any formula of the infinitary logic $\mathscr{L}^\omega_{\infty\omega}(\mathbf{Q})$ on $\mathscr{K}$.*

Proof. The proof is almost identical with that of Proposition 2.18 and is therefore omitted. □

Thus there is no hope of capturing all of PTIME by combining even infinitely many numerical or unary Lindström quantifiers with the infinitary logic $\mathscr{L}^\omega_{\infty\omega}$. Hella [25] proved the stronger result that for any positive integer m there is a polynomial-time property of finite structures which is not expressible in any $\mathscr{L}^\omega_{\infty\omega}(\mathbf{Q})$, where $\mathbf{Q}$ is a possibly infinite sequence of Lindström quantifiers Q_j of type $(n^j_1, \ldots, n^j_l)$ and $n^j_i \leqslant m$ for all $i = 1, \ldots, l$.

3. Pebble games for logics with generalized quantifiers

In algebra the fundamental criterion for distinguishing two structures is whether or not they are isomorphic. From the standpoint of a logic $\mathscr{L}$, two structures are indistinguishable in $\mathscr{L}$ if they satisfy exactly the same sentences of $\mathscr{L}$. This is the key concept for analyzing the expressive power of a logic $\mathscr{L}$ and for comparing it to other logics. In this section we study *equivalence* in the logics $\mathscr{L}^k_{\infty\omega}(\mathbf{Q})$, $k \geqslant 1$, and characterize it in terms of certain infinitary pebble games. For simplicity, we first give the definitions and prove the results for families $\mathbf{Q} = \{Q_i\colon i \in I\}$ of simple unary generalized quantifiers.

Definition 3.1. Let $\mathbf{Q} = \{Q_i\colon i \in I\}$ a sequence of simple unary generalized quantifiers, $\mathbf{A}$ and $\mathbf{B}$ two structures, and k a positive integer.

- Assume that $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ are finite sequences of distinct elements from the universes of $\mathbf{A}$ and $\mathbf{B}$, respectively, where $1 \leqslant m \leqslant k$. We write

$$(\mathbf{A}, a_1, \ldots, a_m) \equiv_{\mathscr{L}^k_{\infty\omega}(\mathbf{Q})} (\mathbf{B}, b_1, \ldots, b_m)$$

to denote that for every formula $\varphi(u_1, \ldots, u_m)$ of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ with free variables among $u_1, \ldots, u_m$ we have that

$$\boldsymbol{A}, a_1, \ldots, a_m \models \varphi(u_1, \ldots, u_m) \text{ if and only if } \boldsymbol{B}, b_1, \ldots, b_m \models \varphi(u_1, \ldots, u_m).$$

- We say that $\boldsymbol{A}$ is *$\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$-equivalent* to $\boldsymbol{B}$, and we write $\boldsymbol{A} \equiv_{\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})} \boldsymbol{B}$, if $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy the same sentences of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$.
- We write $\boldsymbol{A} \equiv_{\mathscr{L}^k_{\infty\omega}} \boldsymbol{B}$ to denote that $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy the same sentences of $\mathscr{L}^k_{\infty\omega}$.

Barwise [7] and Immerman [27] showed that the equivalence relation $\equiv_{\mathscr{L}^k_{\infty\omega}}$ can be characterized in terms of the following *k-pebble game* between two Players I and II on two structures $\boldsymbol{A}$ and $\boldsymbol{B}$. The two players take turns and place pebbles on elements of $\boldsymbol{A}$ and $\boldsymbol{B}$, with Player I choosing first one of the two structures and placing a pebble on an element of it and with Player II responding by placing a pebble on an element of the other structure. Let a_i (b_i) be the element of the structure $\boldsymbol{A}$ (resp. $\boldsymbol{B}$) pebbled in the ith move. After k pebbles have been placed on each structure, if the mapping $a_i \mapsto b_i$, $1 \leqslant i \leqslant k$, is not a *partial isomorphism* between $\boldsymbol{A}$ and $\boldsymbol{B}$ (i.e., an isomorphism between the substructures of $\boldsymbol{A}$ and $\boldsymbol{B}$ generated by the a_i's and the b_i's respectively), then Player I *wins*. Otherwise, Player I removes one pair of corresponding pebbles and the game resumes until again k pebbles have been placed on each structure. We say that Player II *wins the k-pebble game* on $\boldsymbol{A}$ and $\boldsymbol{B}$ if he can continue playing "forever".

Barwise [7] and Immerman [27] showed that $\boldsymbol{A} \equiv_{\mathscr{L}^k_{\infty\omega}} \boldsymbol{B}$ if and only if Player II wins the k-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$. This theorem has become the main technical tool for showing that certain properties are not expressible in the infinitary logic $\mathscr{L}^k_{\infty\omega}$ on finite structures (cf. [27, 34, 9, 30]).

The question that now arises is: can k-pebble games be modified in such a way that the resulting games capture equivalence in the infinitary logics $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, $k \geqslant 1$?

A sufficient, but not necessary, condition for equivalence relative to first-order logic with Lindström quantifiers was given in [49]. A game that captures $\mathscr{L}_{\omega\omega}(\boldsymbol{Q})$-equivalence for monotone simple unary $\boldsymbol{Q}$ was introduced in [37]. A back and forth characterization of $\mathscr{L}_{\omega\omega}(\boldsymbol{Q})$-equivalence for arbitrary $\boldsymbol{Q}$ was given in [10]. A pebble game that captures $\mathscr{L}^k_{\infty\omega}(\boldsymbol{C})$-equivalence was introduced in [9, 30]. We shall introduce next a new pebble game that is capable of capturing $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$-equivalence for arbitrary quantifiers. An inspection of the proof of the theorem of Barwise [7] and Immerman [27] shows that the pebbling of elements of $\boldsymbol{A}$ and $\boldsymbol{B}$ corresponds to finding witnesses for the existential and the universal quantifier. This observation suggests that the first step towards finding games that characterize $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$-equivalence is to allow additional types of moves that correspond to the presence of generalized quantifiers in the logic. The idea is to allow the two players to choose first structures in one of the quantifiers Q_i, $i \in I$, and then to place pebbles on elements of these structures. We formalize this idea by introducing a new pebble game which will then be refined further to yield the desired game that captures equivalence in $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$.

Definition 3.2. Let $\boldsymbol{Q} = \{Q_i: i \in I\}$ be a family of simple unary generalized quantifiers on $\mathscr{K}$, $\boldsymbol{A}$ and $\boldsymbol{B}$ two structures, and k a positive integer. The *$(k, \boldsymbol{Q})$-pebble game* between Players I and II on the structures $\boldsymbol{A}$ and $\boldsymbol{B}$ has the following rules: in each move Player I can play as in the k-pebble game (and Player II must resopond with a move in that game) or Player I can choose one of the structures $\boldsymbol{A}$ and $\boldsymbol{B}$, say $\boldsymbol{A}$, a quantifier Q_j from the family $\boldsymbol{Q}$, and a subset X of the universe A of $\boldsymbol{A}$ such that the structure (A, X) is in the quantifier Q_j. Player II must respond by choosing a subset Y of the universe of the other structure (in this case Y must be a subset of the universe B of $\boldsymbol{B}$) such that the structure (B, Y) is in the quantifier Q_j. Then Player I places a pebble on an element b_1 of $\boldsymbol{B}$ and Player II must respond by placing a pebble on an element a_1 of $\boldsymbol{A}$ such that $a_1 \in X \Leftrightarrow b_1 \in Y$. After this, Player I chooses again one of the two structures and the game continues this way until k pebbles have been placed on each structure. Let a_i and b_i, $1 \leqslant i \leqslant k$, be the elements of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, pebbled by the two players in the ith move. If the mapping $a_i \mapsto b_i$, $1 \leqslant i \leqslant k$, fails to be a partial isomorphism between $\boldsymbol{A}$ and $\boldsymbol{B}$, then Player I wins. Otherwise, Player I removes one pair of corresponding pebbles and the game resumes until k pebbles have been placed on each structure. If the game lasts for infinitely many moves without Player I winning, then Player II is declared the winner.

At first sight, the $(k, \boldsymbol{Q})$-pebble game appears to be the "correct" extension of the k-pebble game and a good candidate for capturing equivalence in the infinitary logics $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, $k \geqslant 1$. It turns out, however, that the $(k, \boldsymbol{Q})$-game is too strong, in the sense that it provides a sufficient condition for $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$-equivalence, but not a necessary one. Indeed, later on we will prove that if Player II wins the $(k, \boldsymbol{Q})$-pebble game on two structures $\boldsymbol{A}$ and $\boldsymbol{B}$, then the two structures satisfy the same sentences of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$. We will also show that the converse may fail for particular simple unary generalized quantifiers. We next introduce a refinement of the $(k, \boldsymbol{Q})$-pebble game in which Player I has less freedom in choosing his moves and, as a result, Player II has a better chance to win.

Definition 3.3. Let $\boldsymbol{Q} = \{Q_i: i \in I\}$ a family of simple unary generalized quantifiers, $\boldsymbol{A}$ and $\boldsymbol{B}$ two structures, and k a positive integer. The *definable $(k, \boldsymbol{Q})$-pebble game* between Players I and II on the structures $\boldsymbol{A}$ and $\boldsymbol{B}$ has the same rules as the $(k, \boldsymbol{Q})$-pebble game with the following exception in the moves of Player I: the sets chosen by Player I must be definable in $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ with the elements of the structures that have pebbles on them as parameters. More specifically, assume it is Player's I turn to make a move and that the corresponding pairs of pebbled elements are $(a_1, b_1), \ldots, (a_m, b_m)$. If Player I chooses one of the two structures, say $\boldsymbol{A}$, a quantifier Q_j in the family $\boldsymbol{Q}$, and a set $X \subseteq A$ such that the structure (A, X) is in Q_j, then there must exist a formula $\varphi(u_1, \ldots, u_m, u)$ of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ with free variables are among $u_1, \ldots, u_m, u$ such that

$$X = \{a \in A: \boldsymbol{A}, a_1, \ldots, a_m, a \models \varphi(u_1, \ldots, u_m, u)\}.$$

The rules for Player II are the same as in the $(k, \boldsymbol{Q})$-pebble game. In particular, Player II is not required to play sets that are definable in $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ with parameters the currently pebbled elements.

We now can state and prove one of the main results of this section.

Theorem 3.4. *Assume that* $\boldsymbol{Q} = \{Q_i\colon i \in I\}$ *is a sequence of simple unary generalized quantifiers,* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are two structures, and* k *is a positive integer. Then the following statements are equivalent*:

(i) $\boldsymbol{A} \equiv_{L^k_{\infty\omega}(\boldsymbol{Q})} \boldsymbol{B}$.

(ii) *Player II has a winning strategy for the definable* $(k, \boldsymbol{Q})$*-pebble game on* $\boldsymbol{A}$ *and* $\boldsymbol{B}$.

If the models $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are finite, we have a third equivalent condition*:

(iii) $\boldsymbol{A} \equiv_{L^k_{\omega\omega}(\boldsymbol{Q})} \boldsymbol{B}$.

Proof. Let us assume first $\boldsymbol{A} \equiv_{L^k_{\infty\omega}(\boldsymbol{Q})} \boldsymbol{B}$. We have to describe the winning strategy of Player II. The strategy is as follows: Suppose pebbles have been put up to now on elements $a_1, \ldots, a_r$ of $\boldsymbol{A}$ and elements $b_1, \ldots, b_r$ of $\boldsymbol{B}$, where $r < k$. During the game Player II will maintain the condition:

$$(\boldsymbol{A}, a_1, \ldots, a_r) \equiv_{L^k_{\infty\omega}(\boldsymbol{Q})} (\boldsymbol{B}, b_1, \ldots, b_r). \tag{1}$$

The strategy of Player II for those moves of Player I that are actually moves in the ordinary k-pebble game is the same as in the ordinary k-pebble game. For details on that we refer to [35, Theorem 2.16]. Let us then assume Player I moves a subset X of, say $\boldsymbol{A}$, so that $(A, X) \in Q_i$ for some $i \in I$. We additionally assume that X is definable by a formula $\varphi(x, z_1, \ldots, z_r)$ of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ with $a_1, \ldots, a_r$ as parameters, i.e.

$$X = \{a \in A\colon \boldsymbol{A}, a, a_1, \ldots, a_r \models \varphi(x, z_1, \ldots, z_r)\}.$$

The strategy of Player II is to play the set

$$Y = \{b \in B\colon \boldsymbol{B}, b, b_1, \ldots, b_r \models \varphi(x, z_1, \ldots, z_r)\}.$$

By applying the equivalence (1) to the formula $Q_i x \varphi(x, z_1, \ldots, z_r)$ we see that $(B, Y) \in Q_i$. Therefore this is a legal move for Player II. Next Player I puts his pebble on an element b of B.

Case 1. $b \in Y$. The strategy of Player II is to put his pebble on an element a of X so that

$$(\boldsymbol{A}, a, a_1, \ldots, a_r) \equiv_{L^k_{\infty\omega}(\boldsymbol{Q})} (\boldsymbol{B}, b, b_1, \ldots, b_r).$$

We claim that this is possible. Indeed, suppose it is not. Then for every $a \in X$ there is a formula $\varphi_a(x, z_1, \ldots, z_r)$ of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ so that $\boldsymbol{A}, a, a_1, \ldots, a_r \models \varphi_a(x, z_1, \ldots, z_r)$ but $\boldsymbol{B}, b, b_1, \ldots, b_r \not\models \varphi_a(x, z_1, \ldots, z_r)$. So

$$\boldsymbol{A}, a_1, \ldots, a_r \models \forall x \left(\varphi(x, z_1, \ldots, z_r) \rightarrow \bigvee_{a \in X} \varphi_a(x, z_1, \ldots, z_r) \right).$$

By (1) again $\boldsymbol{B}, b, b_1, \ldots, b_r \models \bigvee_{a \in X} \varphi_a(x, z_1, \ldots, z_r)$ which contradicts the choice of the formulas $\varphi_a(x, z_1, \ldots, z_r)$. This is the only point, where we need extra care in case we have assumed (iii) only. But if A is finite, then so is X, and the above big disjunction is actually finite.

Case 2. $b \notin Y$: Replace in Case 1 the formula $\varphi(x, z_1, \ldots, z_r)$ everywhere by its negation, and X by its complement.

This ends the description of the winning strategy of Player II.

We shall now assume that Player II has a winning strategy τ and prove $\boldsymbol{A} \equiv_{L^k_{\infty\omega}(Q)} \boldsymbol{B}$. We use induction on the structure of $\varphi(z_1, \ldots, z_r)$ to prove that $\boldsymbol{A}, a_1, \ldots, a_r \models \varphi(z_1, \ldots, z_r)$ if and only $\boldsymbol{B}, b_1, \ldots, b_r \models \varphi(z_1, \ldots, z_r)$, whenever the sequences $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ (without repetitions) represent a pebble position on a round of the game and Player II has been playing the strategy τ. There are different cases to consider. We refer to the proof of [35, Theorem 2.16] for details concerning all but one case, namely, the case that $\varphi(z_1, \ldots, z_r)$ is of the form $Q_i x \psi(x, z_1, \ldots, z_r)$, where x is a variable different from $z_1, \ldots, z_r$. Let us assume $\boldsymbol{A}, a_1, \ldots, a_r \models \varphi(z_1, \ldots, z_r)$. Let

$$X = \{a \in A \colon \boldsymbol{A}, a, a_1, \ldots, a_r \models \psi(x, z_1, \ldots, z_r)\}.$$

Then $(A, X) \in Q_i$. We let Player I play the set X as his next move. The strategy τ directs Player II to play some subset Y of B so that $(B, Y) \in Q_i$. We claim that

$$Y = \{b \in B \colon \boldsymbol{B}, b, b_1, \ldots, b_r \models \psi(x, z_1, \ldots, z_r)\}.$$

Indeed, suppose it is not so.

Case 1. There is some $b \in Y - \{b \in B \colon \boldsymbol{B}, b, b_1, \ldots, b_r \models \psi(x, z_1, \ldots, z_r)\}$: We let Player I put his pebble on this b. The strategy τ directs Player II to put his pebble on some $a \in A$. Since τ is a winning strategy and because of our induction hypothesis, necessarily $a \in X - \{a \in A \colon \boldsymbol{A}, a, a_1, \ldots, a_r \models \psi(x, z_1, \ldots, z_r)\}$, which contradicts the definition of X.

Case 2. There is some $b \in \{b \in B \colon \boldsymbol{B}, b, b_1, \ldots, b_r \models \psi(x, z_1, \ldots, z_r)\} - Y$: The strategy τ directs Player II to put his pebble on some $a \in A$. Necessarily $a \in \{a \in A \colon \boldsymbol{A}, a, a_1, \ldots, a_r \models \psi(x, z_1, \ldots, z_r)\} - X$, which contradicts again the definition of X.

We have proved the claim and $\boldsymbol{B}, b_1, \ldots, b_r \models \varphi(z_1, \ldots, z_r)$ follows. □

Notice that although Player II need not play definable sets in the definable $(k, \boldsymbol{Q})$-pebble game, the above proof shows that he may do so, if he wants to. As a consequence of Theorem 3.4 we obtain the following game-theoretic characterization of definability in the logics $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$, $k \geqslant 1$, for classes of finite structures.

Proposition 3.5. *Let $\boldsymbol{Q} = \{Q_i \colon i \in I\}$ be a family of simple unary generalized quantifiers on the class $\mathscr{F}$ of all finite structures, let $\mathscr{K}$ be a class of finite structures over σ, and let k be a positive integer. Then the following statements are equivalent:*

1. *The class $\mathscr{K}$ is $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$-definable, i.e., there is a sentence φ of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ such that for any finite structure $\boldsymbol{A}$ over σ we have that*

$$\boldsymbol{A} \in \mathscr{K} \iff \boldsymbol{A} \models \varphi.$$

2. *If A and B are finite structures over σ such that $A \in \mathscr{K}$ and Player II has a winning strategy in the definable (k, Q)-pebble game on A and B, then $B \in \mathscr{K}$.*

Proof. Let us suppose condition 2 holds. For any σ-structure A, let φ_A be the conjunction of all $\mathscr{L}^k_{\infty\omega}(Q)$-sentences true in A. Let ψ be the disjunction of all sentences φ_A, where $A \in \mathscr{K}$. If $A \in \mathscr{K}$, then trivially $A \models \psi$. On the other hand, if $B \models \psi$, e.g. $B \models \varphi_A$, where $A \in \mathscr{K}$, then Player II wins the definable (k, Q)-pebble game on A and B. Hence by condition 2, $B \in \mathscr{K}$. □

The preceding results lead to the following method for establishing nonexpressibility results in the infinitary logic $L^{\omega}_{\infty\omega}(Q)$: *In order to prove that a property $\mathscr{P}$ of structures is not expressible by any formula of $L^{\omega}_{\infty\omega}(Q)$, it suffices to show that for any $k \geqslant 1$ there are structures A_k and B_k such that $A_k \models \mathscr{P}$, $B_k \not\models \mathscr{P}$, and Player II has a winning strategy for the definable (k, Q)-pebble game on A_k and B_k.*

Observe that this method is guaranteed to be *complete* by Proposition 3.5, i.e., if the property $\mathscr{P}$ is not expressible in $L^{\omega}_{\infty\omega}(Q)$, then such structures A_k and B_k must exist for every $k \geqslant 1$.

It is obvious that if Player II has a winning strategy for the (k, Q)-pebble game on two structures A and B, then he also has a winning strategy for the definable (k, Q)-pebble game on these two structures. Consequently, the above method can be modified to require that Player II has a winning strategy for the (k, Q)-pebble game on the structures A_k and B_k, for each $k \geqslant 1$. This modified method, however, is not always complete, because there are quantifiers for which the two pebble games are not equivalent (see below).

Although it is easier for Player II to win the definable (k, Q)-pebble game than the (k, Q)-pebble game, in practice it may be hard to describe a winning strategy, because this requires an analysis of $\mathscr{L}^k_{\infty\omega}(Q)$-definability on the structures on which the game is played. Nevertheless, quite often it is possible to describe a winning strategy for Player II in the following *invariant* (k, Q)-game, which is a modification of the other two pebble games and has intermediate strength.

Definition 3.6. Let $Q = \{Q_i: i \in I\}$ be a family of simple unary generalized quantifiers, A and B two structures, and k a positive integer.

1. Assume that $a_1, \ldots, a_m$ are elements of A and X is a subset of A. If $h(X) = X$ for any automorphism h of the structure A such that $h(a_1) = a_1, \ldots, h(a_m) = a_m$, then we say that X is *invariant under automorphisms of A that fix $a_1, \ldots, a_m$*.

2. The *invariant (k, Q)-pebble game* between Players I and II on the structures A and B has the same rules as the (k, Q)-pebble game with the following exception in the moves of Player I: the sets chosen by Player I must be invariant under automorphisms of the structures that fix the currently pebbled elements of the structures. More specifically, assume it is Player's I turn to make a move and that the corresponding pairs of pebbled elements are $(a_1, b_1), \ldots, (a_m, b_m)$. If Player I chooses one of the two structures, say A, a quantifier Q_j in the family Q, and a set $X \subseteq A$ such that the

structure (A, X) is in Q_j, then X must be invariant under automorphisms of A that fix $a_1, \ldots, a_m$. A similar condition must hold if Player I chooses a set $Y \subseteq B$. The rules for Player II are the same as in the $(k, \boldsymbol{Q})$-pebble game. In particular, Player II is not required to play sets that are invariant under such automorphisms.

It is obvious that a winning strategy for Player II in the $(k, \boldsymbol{Q})$-pebble game is also a winning strategy for him in the invariant $(k, \boldsymbol{Q})$-pebble game. The next theorem describes relations between the three pebble games introduced here. Let $Q_{=1/2}$ be the simple unary quantifier consisting of all structures (A, X) such that A is a finite set and X is a subset of A of cardinality $\lfloor |A|/2 \rfloor$, where if x is a nonnegative real number, then $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Theorem 3.7. *Let $\boldsymbol{Q} = \{Q_i\colon i \in I\}$ be a family of simple unary generalized quantifiers, $\boldsymbol{A}$ and $\boldsymbol{B}$ two structures, and k a positive integer.*

- *If Player II has a winning strategy for the invariant $(k, \boldsymbol{Q})$-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$, then he also has a winning strategy for the definable $(k, \boldsymbol{Q})$-pebble on these two structures.*
- *For every positive integer $k \geqslant 1$ there are finite graphs G_k and H_k such that Player II has a winning strategy for the invariant $(k, Q_{=1/2})$-pebble game on G_k and H_k, but Player I has a winning strategy for the $(k, Q_{=1/2})$-pebble game on G_k and H_k.*
- *For every positive integer $k > 2$ there are finite graphs G_k and H_k such that Player II has a winning strategy for the definable $(k, Q_{=1/2})$-pebble game on G_k and H_k, but Player I has a winning strategy for the invariant $(k, Q_{=1/2})$-pebble game (and, hence for the $(k, Q_{=1/2})$-pebble game) on G_k and H_k.*

Proof. Sets which are $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$-definable with parameters are obviously invariant under automorphisms that fix these parameters. This observation yields the *first part* of the proposition.

For a proof of the *second part*, Let G_k consist of cliques C_i of size $2k$ for $1 \leqslant i < 2k$, and H_k of cliques D_i of size $2k$ for $1 \leqslant i < 2k - 2$.

At first we prove that Player I has a winning strategy for the $(k, Q_{=1/2})$-pebble game on G_k and H_k. During the first $k - 1$ rounds of the game Player I puts his pebbles into the cliques C_i, $i < k$. Now comes the $Q_{=1/2}$-move: Player I plays a subset X of G_k which contains $2k - 1$ elements from each C_i for $k \leqslant i < 2k$. So the cardinality of X is $(2k - 1)k = |G_k|/2$ and therefore this is a legal move. Suppose Player II plays a set $Y \subseteq H_k$ of $|H_k|/2 = (2k - 3)k$ elements. Unless Player II has lost the game already, there are but $k - 2$ unpebbled cliques in H_k left. They contain altogether $(2k - 4)k$ elements. Hence Y has to meet a pebbled clique D_i in an element h. Now Player I plays his last pebble on the element h, and wins.

We shall then prove that Player II has a winning strategy for the definable $(k, Q_{=1/2})$-pebble game on G_k and H_k. It is obvious that Player II wins if he can count on Player I never making a $Q_{=1/2}$-move. Hence it suffices to show that Player I cannot play a $Q_{=1/2}$-move, i.e. the quantifier $Q_{=1/2}$ contains no subsets of G_k or H_k definable in $\mathscr{L}^k_{\infty\omega}(Q_{=1/2})$ with $k - 1$ parameters. For this end, suppose X is a subset of

G_k or H_k definable in $\mathscr{L}^k_{\infty\omega}(Q_{=1/2})$ from the parameters $a_1, \ldots, a_r$, where $r < k$. Let $A = \{a_1, \ldots, a_r\}$ and $X' = X - A$. We use the fact that X is closed under automorphisms of the structure that fix the elements $a_1, \ldots, a_r$. There are two useful consequences of this fact. First, if X' meets a clique K, then X contains all elements of $K - A$. Second, if X' meets a clique K with $K \cap A = \emptyset$, then X contains all cliques K with $K \cap A = \emptyset$. These observations help us estimate the size of X.

Case 1: X is a subset of G_k. If X meets only cliques that meet A, then $|X| \leqslant (k-1)2k < |G_k|/2$. If X meets a clique that does not meet A, then $|X| \geqslant 2k^2 > |G_k|/2$.

Case 2: X is a subset of H_k. If X' meets at least $k-1$ cliques, then

$$|X| \geqslant |X'| \geqslant (k-1)2k - (k-1) > 2k^2 - 3k = |H_k|/2.$$

If X' meets at most $k-2$ cliques, then

$$|X| \leqslant |X'| + k - 1 \leqslant (k-2)2k + k - 1 < 2k^2 - 3k = |H_k|/2.$$

In either case X cannot have exactly one half of the elements of the universe.

The proof of the *third part* will be a modification of the proof of the *second part*. The graph G_k consists of cliques C_i of size $2k$ for $1 \leqslant i \leqslant k$ and of cliques D_i of size k for $1 \leqslant i \leqslant 2k$. The graph H_k consists of cliques E_i of size $2k$ for $1 \leqslant i \leqslant k$. Player I wins the invariant $(k, Q_{=1/2})$-pebble game on G_k and H_k as follows. He plays his first $k-1$ pebbles in different cliques C_i and then plays the set $X = \bigcup_{1 \leqslant i \leqslant k} C_i$ of size $2k^2 = |G_k|/2$. Note that X is invariant under automorphisms that fix the pebbled elements. Unless Player II has lost already, he has played his $k-1$ pebbles on different cliques E_i. Now he plays a subset Y of H_k of size $k^2 = |H_k|/2$. The set Y has to contain the elements pebbled by Player II up to now or else he loses immediately. Since $k-1$ sets E_i contribute altogether $(k-1)2k$ elements, Y has to miss an element y from at least one clique E_i which has a pebble in it. Now Player I plays y and wins.

Next we prove that Player II has a winning strategy in the definable $(k, Q_{=1/2})$-pebble game on G_k and H_k. The point is, of course, that the set X played above by Player I is not definable. Let $\boldsymbol{A}$ be either G_k or H_k. Let $a_1, \ldots, a_n$ be elements of $\boldsymbol{A}$ with $n < k$. Let $[a_i]$ denote the clique that a_i is in. Let Z denote the union of the cliques of $\boldsymbol{A}$ which have no pebbles. For any $Y \subseteq \boldsymbol{A}$ let $Y^* = Y - \{a_1, \ldots, a_n\}$.

Claim. *Suppose X is definable in $\mathscr{L}^k_{\infty\omega}$ from the parameters $a_1, \ldots, a_n$. Then X is a union of sets of the form $\{a_i\}$, $[a_i]^*$ and Z^*.*

Proof. An immediate consequence of the fact that X is definable in $\mathscr{L}^k_{\infty\omega}$ from the parameters $a_1, \ldots, a_n$, is that if

$$(\boldsymbol{A}, a_1, \ldots, a_n, x) \equiv_{\mathscr{L}^k_{\infty\omega}} (\boldsymbol{A}, a_1, \ldots, a_n, y)$$

and $x \in X$, then also $y \in X$. Thus if X meets $[a_i]^*$, then $[a_i]^* \subseteq X$, and if X meets Z^*, then $Z^* \subseteq X$. The claim is proved. □

It follows easily from the claim that $|X| \neq |A|/2$. Thus, if $\varphi(a_1, \ldots, a_n, x) \in \mathscr{L}^k_{\infty\omega}$, then

$$\boldsymbol{A} \models Q_{=1/2}x\varphi(a_1, \ldots, a_n, x) \leftrightarrow \exists x \neg x = x.$$

So Player I will not be able to play a definable subset of $\boldsymbol{A}$ of cardinality $|A|/2$. On the other hand, it is obvious that Player II wins the ordinary k-pebble game on G_k and H_k. $\square$

In the sequel we will make heavy use of the invariant $(k, \boldsymbol{Q})$-pebble game in establishing that certain properties $\mathscr{P}$ are not expressible by any formula of $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$ on finite structures. For this, it will be enough to show that for any $k \geqslant 1$ there are structures $\boldsymbol{A}_k$ and $\boldsymbol{B}_k$ such that $\boldsymbol{A}_k \models \mathscr{P}$, $\boldsymbol{B}_k \not\models \mathscr{P}$, and Player II has a winning strategy for the invariant $(k, \boldsymbol{Q})$-pebble game on $\boldsymbol{A}_k$ and $\boldsymbol{B}_k$.

The quantifier $Q_{=1/2}$ used to separate the different versions of $(k, \boldsymbol{Q})$-games, is nonmonotone. This is not an accident, because it turns out that if the quantifiers in $\boldsymbol{Q}$ are monotone, then all the games: $(k, \boldsymbol{Q})$-pebble game, definable $(k, \boldsymbol{Q})$-pebble game, and invariant $(k, \boldsymbol{Q})$-pebble game, are equivalent. As a matter of fact, for monotone quantifiers all three games are equivalent to a somewhat simpler game that we describe next.

Definition 3.8. Let $\boldsymbol{Q} = \{Q_i\colon i \in I\}$ a family of simple unary generalized quantifiers, $\boldsymbol{A}$ and $\boldsymbol{B}$ two structures, and k a positive integer. The *monotone* $(k, \boldsymbol{Q})$-pebble game between Players I and II on the structures $\boldsymbol{A}$ and $\boldsymbol{B}$ has the following rules: in each move Player I can play as in the k-pebble game (and Player II must respond with a move in that game) or Player I can choose one of the structures $\boldsymbol{A}$ and $\boldsymbol{B}$, say $\boldsymbol{A}$, a quantifier Q_j from the family $\boldsymbol{Q}$, and a subset X of the universe A of $\boldsymbol{A}$ such that the structure (A, X) is in the quantifier Q_j. Player II must respond by choosing a subset Y of the universe of the other structure (in this case Y must be a subset of the universe B of $\boldsymbol{B}$) such that the structure (B, Y) is in the quantifier Q_j. Then Player I places a pebble on an element $b_1 \in Y$ and Player II must respond by placing a pebble on an element $a_1 \in X$. After this, Player I chooses again one of the two structures and the game continues this way until k pebbles have been placed on each structure. Let a_i and b_i, $1 \leqslant i \leqslant k$, be the elements of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, pebbled by the two players in the ith move. If the mapping $a_i \mapsto b_i$, $1 \leqslant i \leqslant k$, fails to be a partial isomorphism between $\boldsymbol{A}$ and $\boldsymbol{B}$, then Player I wins. Otherwise, Player I removes one pair of corresponding pebbles and the game resumes until k pebbles have been placed on each structure. If the game lasts for infinitely many moves without Player I winning, then Player II is declared the winner.

Notice that the only difference between the $(k, \boldsymbol{Q})$-pebble game and the monotone $(k, \boldsymbol{Q})$-pebble game is that Player I is restricted to choose elements in the sets played by Player II, while in the $(k, \boldsymbol{Q})$-pebble game he is also allowed to choose elements in the complements of these sets. In general, the monotone $(k, \boldsymbol{Q})$-pebble game is strictly weaker than the $(k, \boldsymbol{Q})$-pebble game, but the two games are equivalent to each other if the quantifiers happen to be monotone.

Theorem 3.9. *Let $\boldsymbol{Q} = \{Q_i : i \in I\}$ a family of monotone simple unary generalized quantifiers, $\boldsymbol{A}$ and $\boldsymbol{B}$ two structures, and k a positive integer. Then the following statements are equivalent:*

(i) $\boldsymbol{A} \equiv_{\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})} \boldsymbol{B}$.

(ii) *Player II has a winning strategy for the $(k, \boldsymbol{Q})$-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$.*

(iii) *Player II has a winning strategy for the monotone $(k, \boldsymbol{Q})$-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$.*

(iv) *Player II has a winning strategy for the definable $(k, \boldsymbol{Q})$-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$.*

(v) *Player II has a winning strategy for the invariant $(k, \boldsymbol{Q})$-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$.*

Furthermore, if the models $\boldsymbol{A}$ and $\boldsymbol{B}$ are finite, we can add another equivalent condition:

(vi) $\boldsymbol{A} \equiv_{\mathscr{L}^k_{\omega\omega}(\boldsymbol{Q})} \boldsymbol{B}$.

Proof. This theorem is proved by establishing (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i). The result then follows from this fact, the preceding Theorem 3.4, and Proposition 3.7.

To prove (i) $\Rightarrow$ (ii), we suppose $\boldsymbol{A} \equiv_{L^k_{\infty\omega}(\boldsymbol{Q})} \boldsymbol{B}$. We have to describe the winning strategy of Player II in the $(k, \boldsymbol{Q})$-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$. The strategy is as follows: Suppose pebbles have been put up to now on elements $a_1, \dots, a_r$ of $\boldsymbol{A}$ and elements $b_1, \dots, b_r$ of $\boldsymbol{B}$, where $r < k$. Part of the strategy is that

$$(\boldsymbol{A}, a_1, \dots, a_r) \equiv_{L^k_{\infty\omega}(\boldsymbol{Q})} (\boldsymbol{B}, b_1, \dots, b_r). \tag{2}$$

Therefore, we need not describe the strategy for those moves of Player I that are actually moves in the ordinary k-pebble game. Let us then assume Player I moves a subset X of, say A, so that $(A, X) \in Q_i$ for some $i \in I$. For any $a \in A$ let $\varphi_a(x, z_1, \dots, z_r) \in L^k_{\infty\omega}(\boldsymbol{Q})$ be the conjunction of all formulas $\varphi(x, z_1, \dots, z_r) \in L^k_{\infty\omega}(\boldsymbol{Q})$ for which $\boldsymbol{A}, a, a_1, \dots, a_r \models \varphi(x, z_1, \dots, z_r)$. Notice that at first sight there is a problem in taking this conjunction, because the collection of formulas of $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ is a proper class. However, by focusing on sets of nonequivalent formulas we can restrict ourselves to a conjunction over a set of formulas. If we assume (vi) rather than (i), we can observe at this point that there are only finitely many formulas of $\mathscr{L}^k_{\omega\omega}(\boldsymbol{Q})$ which are pairwise nonequivalent on $\boldsymbol{A}$. Let X' be the set of elements a of A for which there is an element $a^* \in X$ so that $\boldsymbol{A}, a, a_1, \dots, a_r \models \varphi_{a^*}(x, z_1, \dots, z_r)$. Clearly, $X \subseteq X'$. Therefore $(A, X') \in Q_i$. Note that $a \in X'$ if and only if $\boldsymbol{A}$, $a, a_1, \dots, a_r \models \bigvee_{c \in X} \varphi_c(x, z_1, \dots, z_r)$. The strategy of Player II is to play the set

$$Y = \left\{ b \in B \colon \boldsymbol{B}, b, b_1, \dots, b_r \models \bigvee_{c \in X} \varphi_c(x, z_1, \dots, z_r) \right\}.$$

By our assumption (2), $(B, Y) \in Q_i$. Therefore this is a legal move for Player II. Next Player I puts his pebble on an element b of B. The strategy of Player II is to put his pebble on an element a of A so that $a \in X$ if and only if $b \in Y$ and

$$(A, a, a_1, \ldots, a_r) \equiv_{L^k_{\infty\omega}(Q)} (B, b, b_1, \ldots, b_r). \tag{3}$$

If it happens to be the case that $b \in Y$, then we can pick $a \in X$ so that $B, b, b_1, \ldots, b_r \models \varphi_a(x, z_1, \ldots, z_r)$, and this choice satisfies the condition (3).

Let us then consider the case that $b \notin Y$. Suppose no $a \notin X$ can be found with (3). Then for every $a \notin X'$ there is a formula $\psi_a(x, z_1, \ldots, z_r)$ of $L^k_{\infty\omega}(Q)$ so that

$$A, a, a_1, \ldots, a_r \models \psi_a(x, z_1, \ldots, z_r) \quad \text{and} \quad B, b, b_1, \ldots, b_r \not\models \psi_a(x, z_1, \ldots, z_r).$$

So

$$A, a_1, \ldots, a_r \models \forall x \left(\bigvee_{c \in X} \varphi_c(x, z_1, \ldots, z_r) \vee \bigvee_{a \notin X'} \psi_a(x, z_1, \ldots, z_r) \right).$$

Writing this formula in $\mathscr{L}^k_{\omega\omega}(Q)$ in case we only assumed (vi) is possible, since then X' and X are finite. By (2) again, $B, b, b_1, \ldots, b_r \models \bigvee_{a \notin X'} \psi_a(x, z_1, \ldots, z_r)$ which contradicts the choice of the formulas $\psi_a(x, z_1, \ldots, z_r)$.

This ends the description of the winning strategy of Player II.

To prove (iii) $\Rightarrow$ (i), we shall assume that Player II has a winning strategy τ in the monotone (k, Q)-pebble game on A and B and prove $A \equiv_{L^k_{\infty\omega}(Q)} B$. We use induction on the structure of $\varphi(z_1, \ldots, z_r)$ to prove that $A, a_1, \ldots, a_r \models \varphi(z_1, \ldots, z_r)$ if and only if $B, b_1, \ldots, b_r \models \varphi(z_1, \ldots, z_r)$, whenever the sequences $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ represent a pebble position (without repetitions) on a round of the game and Player II has been playing the strategy τ. The only relevant case here is that $\varphi(z_1, \ldots, z_r)$ is of the form $Q_i x \psi(x, z_1, \ldots, z_r)$, where x is a variable different from $z_1, \ldots, z_r$. Let us assume $A, a_1, \ldots, a_r \models \varphi(z_1, \ldots, z_r)$. Let

$$X = \{a \in A: A, a, a_1, \ldots, a_r \models \psi(x, z_1, \ldots, z_r)\}.$$

Then $(A, X) \in Q_i$. We let Player I play the set X as his next move. The strategy τ directs Player II to play some subset Y of B so that $(B, Y) \in Q_i$. We claim that

$$Y \subseteq \{b \in B: B, b, b_1, \ldots, b_r \models \psi(x, z_1, \ldots, z_r)\}.$$

Suppose not. Then there is some $b \in Y$ with $B, b, b_1, \ldots, b_r \models \neg \psi(x, z_1, \ldots, z_r)$. We let Player I put his pebble on this b. The strategy τ directs Player II to put his pebble on some $a \in A$. Since τ is a winning strategy, we have by the induction hypothesis, $A, a, a_1, \ldots, a_r \models \psi(x, z_1, \ldots, z_r)$, which contradicts the definition of X. We have proved the claim and $B, b_1, \ldots, b_r \models \varphi(z_1, \ldots, z_r)$ follows. □

Elementary equivalence relative to $\mathscr{L}^k_{\infty\omega}(Q)$, where Q is a sequence of arbitrary Lindström quantifiers, can be defined by following the above general guidelines.

Definition 3.10. Let $Q = \{Q_i: i \in I\}$ a family of Lindström quantifiers, A and B two structures, and k a positive integer. Let Q_j be of type $(n^j_1, \ldots, n^j_{l_j})$. The *$(k, Q)$-pebble* *game* between Players I and II on the structures $\boldsymbol{A}$ and $\boldsymbol{B}$ has the following rules: in each move Player I can play as in the k-pebble game (and Player II must respond with a move in that game) or Player I can choose one of the structures $\boldsymbol{A}$ and $\boldsymbol{B}$, say $\boldsymbol{A}$, a quantifier Q_j from the family $\boldsymbol{Q}$, and sets $X_1 \subseteq A^{n_1}, \ldots, X_{l_j} \subseteq A^{n_{l_j}}$ such that the structure $(A, X_1, \ldots, X_{l_j})$ is in the quantifier Q_j. Player II must respond by choosing sets $Y_1 \subseteq B^{n_1}, \ldots, Y_{l_j} \subseteq B^{n_{l_j}}$ such that the structure $(B, Y_1, \ldots, Y_{l_j})$ is in the quantifier Q_j. Then Player I places pebbles on n_i-tuples $\boldsymbol{b}_i$, where $i = 1, \ldots, l_j$, of $\boldsymbol{B}$ and Player II must respond by placing pebbles on n_i-tuples $\boldsymbol{a}_i$, where $i = 1, \ldots, l_j$, of $\boldsymbol{B}$ such that $\boldsymbol{a}_i \in X_i \Leftrightarrow \boldsymbol{b}_i \in Y_i$ for $i = 1, \ldots, l_j$. After this, Player I chooses again one of the two structures and the game continues this way until k pebbles have been placed on each structure. Let a_i and b_i, $1 \leqslant i \leqslant k$, be the elements of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, pebbled by the two players in the ith move. If the mapping $a_i \mapsto b_i$, $1 \leqslant i \leqslant k$, fails to be a partial isomorphism between $\boldsymbol{A}$ and $\boldsymbol{B}$, then Player I wins. Otherwise, Player I removes one pair of corresponding pebbles and the game resumes until k pebbles have been placed on each structure. If the game lasts for infinitely many moves without Player I winning, then Player II is declared the winner. □

It should be clear how *invariant* and *definable* $(k, \boldsymbol{Q})$-pebble game is now defined for Lindström quantifiers $\boldsymbol{Q}$. The proofs of Theorems 3.4 and 3.5 work for Lindström quantifiers with only notational chances. We call a Lindström quantifier Q of type $(n_1, \ldots, n_l)$ *monotone*, if $(A, X_1, \ldots, X_l) \in Q$ and

$$X_1 \subseteq X'_1 \subseteq A^{n_1}, \ldots, X_l \subseteq X'_l \subseteq A^{n_l}$$

imply $(A, X'_1, \ldots, X'_l) \in Q$. Then Theorem 3.9 holds for monotone Lindström quantifiers and its proof needs only notational changes.

4. Structural properties of simple unary generalized quantifiers

In this section we apply the $(k, \boldsymbol{Q})$-pebble games to the study of structural properties of simple unary generalized quantifiers. All structures considered are assumed to be finite. In the first part we study monotone simple unary generalized quantifiers. We show that the unbounded ones are not expressible by the bounded ones and vice versa. We also define what it means for a quantifier to *relativize*, and characterize counting quantifiers as the only monotone simple unary quantifiers which relativize. In the second part we show that a result of Corredor [13] on a subfamily of all simple unary quantifiers can be generalized to the context of infinitary logic with a fixed number of variables.

4.1. *Monotone and counting quantifiers*

In Proposition 2.5 we observed that every simple unary monotone quantifier falls into one of the three categories: eventually counting, eventually bounded, and unbounded. We shall now prove that the first two categories are closed under logical definability and the only one that contains relativizing quantifiers is the first category.

Proposition 4.1. *Suppose Q and Q' are monotone simple unary generalized quantifiers and both nonexpressible in $\mathscr{L}_{\omega\omega}$.*

1. *If Q is eventually bounded and Q' is expressible in $\mathscr{L}^{\omega}_{\infty\omega}(Q)$, then Q' is also eventually bounded.*
2. *If Q is unbounded, $r_Q(n)$ is a monotone function, and Q' is expressible in $\mathscr{L}^{\omega}_{\infty\omega}(Q)$, then Q' is also unbounded.*

Proof. Suppose Q is eventually bounded, that is, there is a positive integer N, and two finite sets $S_1 = \{r_1, \ldots, r_l\}$ and $S_2 = \{s_1, \ldots, s_m\}$ of nonnegative integers such that the following hold:

- S_1 and S_2 are nonempty (the case that one of them is empty is easier).
- For every $n \geqslant N$ there is a j such that either $1 \leqslant j \leqslant l$ and $r_Q(n) = r_j$, or $1 \leqslant j \leqslant m$ and $r_Q(n) = n - s_j$.
- The function $r_Q(n)$ takes each one of the values r_j, $1 \leqslant j \leqslant l$, and $n - s_j$, $1 \leqslant j \leqslant m$, infinitely often.

Towards a contradiction, suppose Q' is an unbounded monotone simple unary quantifier definable in $\mathscr{L}^{k}_{\infty\omega}(Q)$. This implies that there is a positive integer n so that

$$\max\{r_i\colon i = 1, \ldots, l\} + k < r_{Q'}(n) \quad \text{and} \quad \max\{s_i\colon i = 1, \ldots, m\} + k < n - r_{Q'}(n).$$

Let σ be the vocabulary consisting of a unary predicate P. We construct two structures $\boldsymbol{A}$ and $\boldsymbol{B}$ over σ so that the following hold:

(i) Player II wins the invariant (k, Q)-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$.
(ii) $\boldsymbol{A} \in Q'$ and $\boldsymbol{B} \notin Q'$.

The universe of both $\boldsymbol{A}$ and $\boldsymbol{B}$ is $[n]$. In addition, $P^{\boldsymbol{A}} = [r_{Q'}(n)]$ and $P^{\boldsymbol{B}} = [r_{Q'}(n) - 1]$. Condition (ii) is satisfied by construction. To prove condition (i), suppose $r < k$ pebbles have been used and Player I plays an invariant subset X of $\boldsymbol{A}$ or $\boldsymbol{B}$ with $|X| \geqslant r_Q(n)$. Let X' be the part of X which has no pebbles. By considering separately the cases that X' meets P and its complement, X' meets only P, X' meets only the complement of P, and $X' = \emptyset$, one shows easily that Player II can choose $Y \subset B$ with $|Y| \geqslant r_Q(n)$ so that whatever $y \in Y$ Player I pebbles, Player II can find $x \in X$ preserving the partial isomorphism property. This ends the proof of 1.

The proof of 2 is similar. □

Proposition 4.2. *Suppose Q is a monotone simple unary quantifier and Q is expressible in $\mathscr{L}^{\omega}_{\infty\omega}$. Then Q is eventually counting.*

Proof. The proof is similar to the proof of Proposition 4.1. □

Corollary 4.3. *Suppose Q is a monotone simple unary quantifier. Then the following conditions are equivalent:*

(i) *Q is expressible in $\mathscr{L}_{\omega\omega}$.*
(ii) *Q is expressible in $\mathscr{L}^{\omega}_{\infty\omega}$.*
(iii) *Q is eventually counting.*

Intuitively speaking a quantifier has the relativization property, if whatever it says of the universe, it can also say of the restriction of the universe into a unary predicate. For example, suppose we have the quantifier $Q = \{(A, X): X \subseteq A, |X| \geqslant |A|/2\}$. With this quantifier we can say things like "At least half of the vertices are colored red and have degree 3". If this quantifier was relativizing, we could also say things like "At least half of red vertices have degree 3".

Suppose σ is a vocabulary with no constant symbols and no function symbols. Suppose A is structure over σ and $P \in \sigma$ is a unary predicate. The *relativization* A^P of A to P is the substructure of A the universe of which is the interpretation of P in A.

Definition 4.4. Let $\mathbf{Q}$ be a (finite or infinite) sequence of generalized quantifiers. We say that $\mathscr{L}^k_{\infty\omega}(\mathbf{Q})$ has the *relativization property* if for any vocabulary σ with no constant symbols and no function symbols, any unary predicate $P \in \sigma$ and any $\varphi \in \mathscr{L}^k_{\infty\omega}(\mathbf{Q})$ over σ there is $\varphi^P \in \mathscr{L}^k_{\infty\omega}(\mathbf{Q})$ so that for A over σ

$$A \models \varphi^P \iff A^P \models \varphi.$$

Relativization property for $\mathscr{L}^{\omega}_{\infty\omega}(\mathbf{Q})$ is defined similarly.

Example 4.5. The logic $\mathscr{L}^k_{\infty\omega}(\mathbf{C})$ has the relativization property. This can be proved by induction on the complexity of formulas. The crucial step in the induction is the definition

$$(\exists i x \varphi(x))^P = \exists i x (P(x) \wedge \varphi(x)).$$

Example 4.6. The logic $\mathscr{L}^k(Q_{\text{even}})$ has the relativization property. The crucial step in the induction is

$$(Q_{\text{even}} x \varphi(x))^P = Q_{\text{even}} x (P(x) \wedge \varphi(x)).$$

Proposition 4.7. *Suppose Q is a monotone simple unary quantifier. Then $\mathscr{L}^k_{\infty\omega}(Q)$ has the relativization property if and only if Q is eventually counting.*

Proof. First of all, if Q is eventually counting, it is easy to see that $\mathscr{L}^k_{\infty\omega}(Q)$ has the relativization property. Let us then assume $\mathscr{L}^k_{\infty\omega}(Q)$ has the relativization property. If Q is not eventually counting, it is either unbounded or eventually bounded. Suppose first Q is unbounded. Let σ be a vocabulary with two predicate symbols P_1 and P_2.

Choose m so that $min(r_Q(m), m - r_Q(m)) > k$. Let n be a positive integer so that $min(r_Q(n), n - r_Q(n)) > m + k$. Let $\boldsymbol{A}$ be a structure over σ with $[n]$ as universe and with $P_1^{\boldsymbol{A}} = \{1, \ldots, k\}$, $P_2^{\boldsymbol{A}} = \{1, \ldots, m\}$. Respectively, let $\boldsymbol{B}$ be a structure over σ with $\{1, \ldots, n\}$ as universe and with $P_1^{\boldsymbol{B}} = \{1, \ldots, r_Q(m)\}$, $P_2^{\boldsymbol{B}} = \{1, \ldots, m\}$. Now

$$\boldsymbol{A} \nvDash (QxP_1(x))^{(P_2)} \quad \text{and} \quad \boldsymbol{B} \vDash (QxP_1(x))^{(P_2)},$$

so it remains to show that Player II wins the invariant (k, Q)-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$. For this end, suppose we are in the middle of the game, and pebbles have been placed on elements $a_1, \ldots, a_r$ of $\boldsymbol{A}$ and $b_1, \ldots, b_r$ of $\boldsymbol{B}$, where $r < k$. Suppose Player I plays now a subset X of, say $\boldsymbol{A}$, with $|X| \geqslant r_Q(n)$. Let us assume, for simplicity, that $X \cap \{a_1, \ldots, a_r\} = \{a_1, \ldots, a_s\}$. Since the predicate P_2 has less than $r_Q(n) - k$ elements, X contains an element c outside $P_2 \cup \{a_1, \ldots, a_r\}$. The strategy of Player II is to let his set Y consist of elements outside P_2 plus the elements $\{b_1, \ldots, b_s\}$. Since there are at least $k + r_Q(n)$ elements outside P_2, Player II can make sure that $|Y| \geqslant r_Q(n)$. Next Player I puts a pebble on an element y of Y. If $y \in \{b_1, \ldots, b_s\}$, Player II puts his pebble on the corresponding element of $\{a_1, \ldots, a_s\}$. If, on the other hand, y is outside P_2, Player II puts his pebble on c. The case that X is a subset of $\boldsymbol{B}$ is entirely similar. This ends the description of the winning strategy of Player II.

Suppose then Q is eventually bounded. Let N, $S_1 = \{r_1, \ldots, r_l\}$, $S_2 = \{s_1, \ldots, s_m\}$ be as in the definition of eventual boundedness (Definition 2.4).

Case 1: *Both S_1 and S_2 are nonempty*. Choose $m_1 > 2k + s_1$ so that $r_Q(m_1) = m_1 - s_1$, and $m_2 > 2k + r_1$ so that $r_Q(m_2) = r_1$. Let $n > m_1 + m_2 + k + s_1$ be such that $r_Q(n) = n - s_1$. Let $\boldsymbol{A}$ be a structure over σ with $[n]$ as universe and with $P_1^{\boldsymbol{A}} = [k]$, $P_2^{\boldsymbol{A}} = [m_1]$. Respectively, let $\boldsymbol{B}$ be a structure over σ with $[n]$ as universe and with $P_1^{\boldsymbol{B}} = [k + r_1]$, $P_2^{\boldsymbol{B}} = [m_2]$. Now

$$\boldsymbol{A} \nvDash (QxP_1(x))^{(P_2)} \quad \text{and} \quad \boldsymbol{B} \vDash (QxP_1(x))^{(P_2)},$$

so it remains to show that Player II wins the invariant (k, Q)-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$. For this end, suppose we are in the middle of the game, and pebbles have been placed on elements $a_1, \ldots, a_r$ of $\boldsymbol{A}$ and $b_1, \ldots, b_r$ of $\boldsymbol{B}$, where $r < k$. Suppose Player I plays now a subset X of, say $\boldsymbol{A}$, with $|X| \geqslant r_Q(n) = n - s_1$. Let us assume, for simplicity, that $X \cap \{a_1, \ldots, a_r\} = \{a_1, \ldots, a_s\}$. Since the predicate P_2 has less than $r_Q(n) - k$ elements, X contains an element c outside $P_2 \cup \{a_1, \ldots, a_r\}$. The strategy of Player II is to let his set Y consist of elements outside P_2 plus the elements $\{b_1, \ldots, b_s\}$. Since there are at least $r_1 + k$ elements outside P_2, Player II can make sure that $|Y| \geqslant r_Q(n)$. Next Player I puts a pebble on an element y of Y. If $y \in \{b_1, \ldots, b_s\}$, Player II puts his pebble on the corresponding element of $\{a_1, \ldots, a_s\}$. If, on the other hand, y is outside P_2, Player II puts his pebble on c. The case that X is a subset of $\boldsymbol{B}$ is entirely similar. This ends the description of the winning strategy of Player II.

Case 2: *$S_1 \neq \emptyset$ and $S_2 = \emptyset$*. By assumption, S_1 has at least two elements r_1 and $r_2 > r_1$. Choose $m_1 > k + r_1 + r_2 + 1$ so that $r_Q(m_1) = r_1$, and $m_2 > k + r_1 + r_2 + 1$ so that $r_Q(m_2) = r_2$. Let $n > m_1 + m_2 + k + 1$ be such that $r_Q(n) = r_1$. Let $\boldsymbol{A}$ be a structure over σ with $[n]$ as universe and with $P_1^A = [r_1]$, $P_2^A = [m_1]$. Respectively, let $\boldsymbol{B}$ be a structure over σ with $[n]$ as universe and with $P_1^B = [r_2 - 1]$, $P_2^B = [m_2]$. Now

$$\boldsymbol{A} \models (QxP_1(x))^{(P_2)} \quad \text{and} \quad \boldsymbol{B} \not\models (QxP_1(x))^{(P_2)},$$

so it remains to show that Player II wins the invariant (k, Q)-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$. For this end, suppose we are in the middle of the game, and pebbles have been placed on elements $a_1, \ldots, a_r$ of $\boldsymbol{A}$ and $b_1, \ldots, b_r$ of $\boldsymbol{B}$, where $r < k$. Suppose Player I plays now a subset X of, say $\boldsymbol{A}$, with $|X| \geqslant r_Q(n) = r_1$. Let us assume, for simplicity, that $X \cap \{a_1, \ldots, a_r\} = \{a_1, \ldots, a_s\}$. Let $X' = X - \{a_1, \ldots, a_s\}$. If X' meets $P_2^A - P_1^A$, Player II lets his set Y consist of $\{b_1, \ldots, b_s\}$ plus $r_1 - s$ elements from $P_2^B - (P_1^B \cap \{b_1, \ldots, b_r\})$. This is possible, since this set has at least $m_2 - r_1 + 1 - k > r_1$ elements. If X' meets $[n] - P_2^A$, Player II lets his set Y consist of $\{b_1, \ldots, b_s\}$ plus $r_1 - s$ elements from $[n] - (P_2^B \cap \{b_1, \ldots, b_r\})$. This is possible, since this set has at least $n - m_2 - k > m_1 + 1 > r_1$ elements. If X' is contained in P_1^A and $X' \neq \emptyset$, then $|X| = r_1 + t$, where t is the number of elements from $\{a_1, \ldots, a_s\}$ which are outside P_1^A. In this case Player II lets Y consist of $r_1 - s$ elements from P_1^B plus the elements $\{b_1, \ldots, b_r\}$. Finally, if $X' = \emptyset$, Y is chosen to be $\{b_1, \ldots, b_r\}$.

Next Player I put a pebble on an element y of Y. By going through the above different possibilities for the choice of Y, one can be convinced that a successful choice of $x \in X$ for Player II has been guaranteed in each case. This ends the description of the winning strategy of Player II.

Case 3: $S_1 = \emptyset$ *and* $S_2 \neq \emptyset$. This case is similar to Case 2, only slightly easier. Therefore we omit the details. □

Independently of us, Westerståhl [51] proved Proposition 4.7 for $\mathscr{L}_{\omega\omega}(\boldsymbol{Q})$. Flum [17, Theorem 4.1] has a characterization of relativizing monotone simple unary quantifiers on infinite structures.

4.2. Universe-independent quantifiers

In general, the truth of a sentence of the form $Qx\varphi(x, \boldsymbol{a})$ in a model $\boldsymbol{A}$ depends on the cardinality of the domain of $\boldsymbol{A}$. This is the case, for example, with Q_{half} but not with Q_{even}. Let us call a quantifier *universe-independent* (following [51]) if its definition is independent of the cardinality of the domain of the model, that is, $m \in f_Q(n)$ for some $n \geqslant m$ if and only if $m \in f_Q(n)$ for all $n \geqslant m$. The universe-independent quantifiers are a special case of relativizing quantifiers. Except in trivial cases, they are non-monotone.

Corredor [13] investigates the following class of simple unary generalized quantifiers: If S is a set of natural numbers, we define

$$C_S = \{(A, X) \colon X \subseteq A, |X| \in S\}.$$

These quantifiers are universe-independent and, indeed, every simple unary universe-independent quantifier is of this form.

Corredor gives a characterization of universe-independent quantifiers definable in $\mathscr{L}_{\omega\omega}(C_{S_1}, \ldots, C_{S_n})$. We shall elaborate Corredor's proof and get a characterization of universe-independent quantifiers definable in $\mathscr{L}^{\omega}_{\infty\omega}(C_{S_1}, \ldots, C_{S_n})$ on finite structures.

If A is a set of natural numbers and n is a natural number, we let $A \oplus m = \{i + m \mid i \in A\}$. Note that $C_{S \oplus m}$ is definable in $\mathscr{L}_{\omega\omega}(C_S)$. Let $S \sim S'$ if $S \triangle S'$ is finite. Clearly, if $S \sim S'$, then C_S is definable in $\mathscr{L}_{\omega\omega}(C_{S'})$. Lemma 4.8 and Proposition 4.9 are essentially contained in [13]. We give a proof for completeness.

Lemma 4.8 (Corredor [13]). *Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are models of the empty vocabulary and both have cardinality $\geqslant k$. Suppose for all $m < k$:*

$$|\boldsymbol{A}| \in S \oplus m \iff |\boldsymbol{B}| \in S \oplus m.$$

Then Player II wins the invariant (k, C_S)-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$.

Proof. Let us assume, for simplicity, that all k pebbles have been played, but Player II has not lost yet. Suppose I picks a pebble and a subset X of one of the models, say $\boldsymbol{A}$, so that $|X| \in S$.

Case 1: X is a set of pebbled elements. Let Y be the set of corresponding pebbled elements in $\boldsymbol{B}$. Certainly $|Y| \in S$. Suppose now I puts a pebble on an element y of $\boldsymbol{B}$ not in Y. If y was not one of the already pebbled elements, II uses the fact that $\boldsymbol{A}$ has more than $k - 1$ elements to find a matching element.

Case 2: X contains an unpebbled element. Now X is actually the complement of a set of, say m, pebbles, where $m < k$. Thus $|\boldsymbol{A}| \in S \oplus m$, whence $|\boldsymbol{B}| \in S \oplus m$. So the size of the complement Y of the corresponding set of pebbled elements in $\boldsymbol{B}$ is in S. Thus II can play this set and stay in the game. □

Proposition 4.9 (Corredor [13]). *Let $S, S_1, \ldots, S_n$ be sets of integers. Then the quantifier C_S is definable in $\mathscr{L}^{\omega}_{\infty\omega}(C_{S_1}, \ldots, C_{S_n})$ if and only if there are numbers $m_1, \ldots, m_j, n_1, \ldots, n_j$ and a Boolean combination S' of the sets $S_{n_1} \oplus m_1, \ldots, S_{n_j} \oplus m_j$ so that $S \sim S'$.*

Proof. Suppose C_S is definable in $\mathscr{L}^{\omega}_{\infty\omega}(C_{S_1}, \ldots, C_{S_n})$. It follows that some identity-sentence φ in $\mathscr{L}^{k}_{\omega\omega}(C_{S_1}, \ldots, C_{S_n})$ is equivalent to $C_S x(x = x)$. Suppose also that $S \nsim S'$ if S' is a Boolean combination of the sets $S_i \oplus m$, where $m \leqslant k$. It follows that there are numbers $u \geqslant k$ and $v \geqslant k$ such that $u \in S$, $v \notin S$ but

$$u \in S_i \oplus m \iff v \in S_i \oplus m$$

for all $i \in \{1, \ldots, n\}$ and $m < k$. To get a contradiction, let $\boldsymbol{A}$ be a structure of cardinality u for the empty vocabulary and $\boldsymbol{B}$ similarly a structure of cardinality v. Now $\boldsymbol{A} \models \varphi$, $\boldsymbol{B} \models \neg\varphi$ and II wins the $(k, C_{S_1}, \ldots, C_{S_n})$-game on $\boldsymbol{A}$ and $\boldsymbol{B}$. □

Proposition 4.9 gives a relatively simple method for deciding whether an universe-independent quantifier is definable from other universe-independent quantifiers using $\mathscr{L}^{\omega}_{\infty\omega}$. Of course, universe-independent quantifiers constitute a very special category of simple unary quantifiers. But it is interesting to note that in this category one obtains a complete picture of mutual definability relations.

5. Finitely many versus infinitely many simple unary quantifiers

In this section we consider finite models only. We have already observed in Proposition 2.7 that every unary quantifier can be defined in terms of the infinitely many counting quantifiers and $\mathscr{L}^{\omega}_{\infty\omega}$. This raises the question:

> Is there a finite sequence $\boldsymbol{Q}$ of simple unary generalized quantifiers such that every simple unary quantifier is definable in $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$?

A negative answer to this question can be obtained with a diagonal argument. Indeed, we show in Proposition 5.1 that for every finite sequence $\boldsymbol{Q}$ of simple unary quantifiers there is a property expressible in $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{C})$, but not in $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$. After this easy answer we pose a new question:

> Is there a property of finite structures that is expressible in $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{C})$, but not expressible in $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$ for any finite sequence $\boldsymbol{Q}$ of simple unary quantifiers?

The main results of this chapter show that this question can be answered affirmatively. More specifically, in Theorems 5.3 and 5.8 we show that there are natural polynomial-time properties that are expressible in $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{C})$, but are not expressible in $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$ for any finite sequence of simple unary generalized quantifiers. For this, we introduce first the $(k, \boldsymbol{Q})$*-coloring method*, which, intuitively, classifies subsets of potential structures according to what the quantifiers $\boldsymbol{Q}$ can say about them. The proofs of the main results are then obtained by combining the invariant $(k, \boldsymbol{Q})$-pebble games, the $(k, \boldsymbol{Q})$-coloring method, and a Ramsey-theoretic model construction.

Suppose $\boldsymbol{Q} = (Q_1, \ldots, Q_m)$ is a sequence of simple unary generalized quantifiers and k is a positive integer. The number k and the sequence $\boldsymbol{Q}$ impose a natural coloring $\eta_{k,\boldsymbol{Q}}$ on positive integers as follows: Let f_i be the defining function of Q_i. We put $\eta_{k,\boldsymbol{Q}}(a) = \eta_{k,\boldsymbol{Q}}(b)$ if and only if the following conditions hold:

(C1) If $a < k$ or $b < k$, then $a = b$.
(C2) $s \in f_i(a) \Leftrightarrow s \in f_i(b)$, whenever $1 \leqslant i \leqslant m$ and $0 \leqslant s \leqslant k$.
(C3) $a - s \in f_i(a) \Leftrightarrow b - s \in f_i(b)$, whenever $1 \leqslant i \leqslant m$ and $0 \leqslant s \leqslant k$.

This mapping colors the set of positive integers with at most $4^{(k+1)m} + k$ colors.

Proposition 5.1. *For any finite sequence $\boldsymbol{Q} = (Q_1, \ldots, Q_m)$ of simple unary quantifiers on $\mathscr{F}$ there is a simple unary quantifier Q which is not expressible in $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$. In particular, $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q}) \neq \mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{C})$. Moreover, if each Q_i is PTIME, then so is Q.*

Proof. It is easy to see, that if $\eta_{k,Q}(a) = \eta_{k,Q}(b)$ and $a, b \geqslant k$, then Player II wins the definable $(k, \boldsymbol{Q})$-pebble game on the structures $([a])$ and $([b])$ over the empty vocabulary. In particular, for such a and b we have $([a]) \equiv_{\mathscr{L}^k_{\infty\omega}(Q)} ([b])$. Let $v_0 = 0$. For $k \geqslant 1$, let u_k be the least integer $\geqslant v_{k-1}$ such that there is a least $v_k > u_k$ such that $\eta_{k,Q}(u_k) = \eta_{k,Q}(v_k)$. We define a simple unary quantifier Q be letting $f_Q(v_k) = \{v_k\}$ for all $k \geqslant 1$, and $f_Q(n) = \emptyset$ otherwise. Now Q is not definable in $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{Q})$, since for each $k \geqslant 1$ we have $([u_k]) \not\equiv_{L^1_{\omega\omega}(Q)} ([v_k])$, but $([u_k]) \equiv_{\mathscr{L}^k_{\infty\omega}(Q)} ([v_k])$. Suppose then each quantifier in $\boldsymbol{Q}$ is PTIME. We describe a polynomial-time algorithm for deciding whether a given model of size n is in Q or not. It suffices to decide whether $n = v_k$ for some $k \leqslant n$. This problem reduces to the problem of deciding whether two nonnegative integers $\leqslant n$ have the same $\eta_{k,Q}$-color or not for any given $k \leqslant n$. This problem is solved for integers $\geqslant k$ by deciding the $(k + 1)m$ questions "$s \in f_i(a)$?" and the $(k + 1)m$ questions "$a - s \in f_i(a)$?", and comparing the $2(k + 1)m$ answers. □

Suppose again that $\boldsymbol{Q} = (Q_1, \ldots, Q_m)$ is a sequence of simple unary generalized quantifiers and k and n are positive integers. The numbers k and n and the sequence $\boldsymbol{Q}$ impose a coloring $\chi_{k,Q}$ on the elements of $\{1, \ldots, n\}$ as follows: Let f_i be the defining function of Q_i. We put $\chi_{k,Q}(a) = \chi_{k,Q}(b)$ if and only if the following conditions hold:

(D1) If $a < k$ or $b < k$ or $a > n - k$ or $b > n - k$, then $a = b$.
(D2) $a + s \in f_i(n) \Leftrightarrow b + s \in f_i(n)$, whenever $1 \leqslant i \leqslant m$ and $-k < s < k$.
(D3) $n - a - s \in f_i(n) \Leftrightarrow n - b - s \in f_i(n)$, whenever $1 \leqslant i \leqslant m$ and $-k < s < k$.

We call this coloring the *$(k, \boldsymbol{Q})$-coloring* of $\{1, \ldots, n\}$. As a first primitive application of the $(k, \boldsymbol{Q})$-coloring method, we have the following proposition.

Proposition 5.2. *Given a number $k \geqslant 1$ and a finite sequence $\boldsymbol{Q} = (Q_1, \ldots, Q_m)$ of simple unary generalized quantifiers on $\mathscr{F}$, there is a counting quantifier that is not expressible in $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$.*

Proof. Notice first that the $(k, \boldsymbol{Q})$-coloring partitions $\{1, \ldots, n\}$ into at most $4^{(2k-1)m} + 2k$ classes. Let $n > 4^{(2k-1)m} + 2k$. By the pigeon-hole principle there are two distinct elements a and b of $\{1, \ldots, n\}$ with the same color with $a < b$. We show that the counting quantifier $(\exists a\, x)$ is not expressible in $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$. For this end we define two models $\boldsymbol{A}$ and $\boldsymbol{B}$ of the vocabulary consisting of one unary predicate symbol P only. The universe of both models is $[n]$. Moreover, $P^A = [a]$ and $P^B = [b]$. So the sentence $\exists b\, x P(x)$ holds in $\boldsymbol{A}$ but not in $\boldsymbol{B}$. It remains to show that Player II wins the $(k, \boldsymbol{Q})$-pebble game. This easy argument is left to the reader. □

In the sequel, we shall use more sophisticated combinatorial techniques to get elements that not only have the same color, but also satisfy some further properties useful in the definable $(k, \boldsymbol{Q})$-pebble game. Note that this method of constructing similar, but nonisomorphic, structures relies heavily on the finiteness of the sequence $\boldsymbol{Q}$. For equivalence relative to infinitely many quantifiers completely different methods have to be used, like those used by [9, 25]. On the other hand, the models we construct could not possibly be equivalent relative to all counting quantifiers.

The *Härtig* quantifier was defined earlier in Section 2.5. This quantifier is readily expressible in $\mathscr{L}^1_{\infty\omega}(\boldsymbol{C})$, since the formula $Ix, y(\varphi(x), \psi(y))$ is equivalent to $\bigvee_n((\exists!nx)\varphi(x) \wedge (\exists!nx)\varphi(x))$, where $(\exists!ix)\theta$ is short for $(\exists ix)\theta \wedge \neg(\exists i+1x)\theta$. We may conclude that although I itself is nonsimple, it can be expressed, as any numerical quantifier (Proposition 2.18) and any unary quantifier (Proposition 2.20), with an infinite set of simple unary quantifiers. We next establish an optimal lower bound for the expressibility of the Härtig quantifier.

Theorem 5.3. *Suppose $\boldsymbol{Q}$ is a finite sequence of simple unary generalized quantifiers on $\mathscr{F}$. Then the Härtig quantifier I is not expressible in $\mathscr{L}^\omega_{\infty\omega}(\boldsymbol{Q})$.*

Proof. Let σ be the vocabulary consisting of three unary predicates P_1, P_2 and P_3. We construct two structures $\boldsymbol{A}$ and $\boldsymbol{B}$ over σ so that the following hold:
- Player II wins the invariant $(k, \boldsymbol{Q})$-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$.
- $\boldsymbol{A} \not\equiv_{L^1_{\omega\omega}(I)} \boldsymbol{B}$.

We shall use the following well-known theorem.

Van der Waerden's Theorem. *For all positive integers k and r there exists an integer $W(k,r)$ such that if the set of integers $\{1, \ldots, W(k,r)\}$ is partitioned into r classes, then at least one class contains a k-term arithmetic progression*

$$a, a+d, a+2d, \ldots, a+kd.$$

(For a proof, see e.g. [20].)

Let $n = 2W(2, 4^{(2k-1)m} + 2k)$. We let the universe of both $\boldsymbol{A}$ and $\boldsymbol{B}$ be the set $[n]$. By applying van der Waerden's Theorem to the $(k, \boldsymbol{Q})$-coloring $\chi_{k,\boldsymbol{Q}}$, we can choose positive integers a and d so that the numbers $a, a+d, a+2d$ are all of the same color. Let

$$P_1^A = [a+d], \qquad P_2^A = [a+d+1, 2(a+d)],$$
$$P_3^A = [2(a+d)+1, n], \qquad P_1^B = [a],$$
$$P_2^B = [a+1, 2(a+d)], \qquad P_3^B = [2(a+d)+1, n].$$

Note that $|P_1^B| = |P_2^B|$, but $|P_1^A| \neq |P_2^A|$. So we only have to prove that II wins the definable $(k, \boldsymbol{Q})$-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$. Let us suppose r pebbles ($r < k$) have been played and it is I's turn to move. Player I chooses a subset X of one of the models, with $([n], X) \in Q_j$. We use the fact that X is invariant under all automorphisms of the model which fix the pebbled elements. Let the pebbled elements of $\boldsymbol{A}$ be $a_1, \ldots, a_r$ and let the corresponding elements of $\boldsymbol{B}$ be $b_1, \ldots, b_r$. We assume as an induction hypothesis that the mapping $a_i \mapsto b_i$ is a partial isomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. Let

$A_0 = \{a_1, \ldots, a_r\}$ and $B_0 = \{b_1, \ldots, b_r\}$. Let us assume X is a subset of the domain of $\boldsymbol{A}$. The case that X is a subset of the domain of $\boldsymbol{B}$ is entirely similar. Let $X' = X - A_0$.

Case 1: X' does not meet any of the sets P_i^A. In this case $X \subseteq A_0$ and Player II chooses as his set Y the corresponding subset of B_0. Since $|X| = |Y|$, we have $([n], Y) \in Q_j$. Next Player I can put a pebble on an element $b_i \in Y$ and then Player II responds by putting a pebble on $a_i \in X$. On the other hand, Player II can choose to put a pebble on $b \notin B_0$. So $b \notin Y$. Say, $b \in P_i^B$. Since $|P_i^A| \geqslant k$, Player II can put a pebble on an element of $|P_i^A| - A_0$.

Case 2: X' meets exactly one of the sets P_i^A.

Subcase 2.1: X' meets P_3^A. Using automorphisms, it can be seen that $X = (P_3^A - A_1) \cup A_2$, where $\{A_1, A_2\}$ is a partition of A_0. Let $\{B_1, B_2\}$ be the corresponding partition of B_0 and $Y = (P_3^B - B_1) \cup B_2$. Since $|X| = |Y|$, we have $([n], Y) \in Q_j$. It is clear how Player II continues from here.

Subcase 2.2: X' meets P_1^A. Then $X = (P_1^A - A_1) \cup A_2$, where $\{A_1, A_2\}$ is a partition of A_0. Let $\{B_1, B_2\}$ be the corresponding partition of B_0 and $Y = (P_1^B - B_1) \cup B_2$. Now

$$|X| = a + d - |A_1| + |A_2|, \qquad |Y| = a - |B_1| + |B_2|.$$

Since $|A_1| = |B_1|$, $|A_2| = |B_2|$ and $\chi_{k,Q}(a) = \chi_{k,Q}(a + d)$, we have $([n], Y) \in Q_j$.

Subcase 2.3: X' meets P_2^A. Then $X = (P_2^A - A_1) \cup A_2$, where $\{A_1, A_2\}$ is a partition of A_0. Let $\{B_1, B_2\}$ be the corresponding partition of B_0 and $Y = (P_2^B - B_1) \cup B_2$. Now

$$|X| = a + d - |A_1| + |A_2| \text{ and } |Y| = a + 2d - |B_1| + |B_2|.$$

Since $\chi_{k,Q}(a + d) = \chi_{k,Q}(a + 2d)$, we have $([n], Y) \in Q_j$.

Case 3: X' meets exactly two of the sets P_i^A.

Subcase 3.1: X' meets P_1^A and P_3^A. Then $X = (([n] - P_2^A) - A_1) \cup A_2$, where $\{A_1, A_2\}$ is a partition of A_0. Let $\{B_1, B_2\}$ be the corresponding partition of B_0 and $Y = (([n] - P_2^B) - B_1) \cup B_2$. Now

$$|X| = n - (a + d) - |A_1| + |A_2| \quad \text{and} \quad |Y| = n - (a + 2d) - |B_1| + |B_2|.$$

Since $\chi_{k,Q}(a) = \chi_{k,Q}(a + d)$, we have $([n], Y) \in Q_j$.

Subcase 3.2: X' meets P_2^A and P_3^A. Then $X = (([n] - P_1^A) - A_1) \cup A_2$, where $\{A_1, A_2\}$ is a partition of A_0. Let $\{B_1, B_2\}$ be the corresponding partition of B_0 and $Y = (([n] - P_1^B) - B_1) \cup B_2$. Now

$$|X| = n - a - |A_1| + |A_2| \quad \text{and} \quad |Y| = n - (a + 2d) - |B_1| + |B_2|.$$

Since $\chi_{k,Q}(a) = \chi_{k,Q}(a + 2d)$, we have $([n], Y) \in Q_j$.

Subcase 3.3: X' meets P_1^A and P_2^A. Then $X = ((P_1^A \cup P_2^A) - A_1) \cup A_2$, where $\{A_1, A_2\}$ is a partition of A_0. Let $\{B_1, B_2\}$ be the corresponding partition of B_0 and $Y = ((P_1^B \cup P_2^B) - B_1) \cup B_2$. Now $|X| = |Y|$, so we have $([n], Y) \in Q_j$.

Case 4: X' meets each P_i^A. In this case X is the complement of a subset of A_0 and Player II chooses as his set Y the complement of the corresponding subset of B_0. Since $|X| = |Y|$, we have $([n], Y) \in Q_j$.

We have demonstrated how Player II continues to maintain $a_i \mapsto b_i$ as a partial isomorphism in each of the four cases. □

The theorem remains true if Härtig quantifier is replaced by the quantifier $\{(A, X, Y): X, Y \subseteq A, |A| = m|B|\}$, where m is an arbitrary but fixed positive integer. For this and other similar extensions of the above theorem, see [46]. For another application of van der Waerden's Theorem in generalized quantifiers, see [31].

Corollary 5.4. *The queries "do two given vertices have the same degree?" and "does a given graph have two connected components of the same size?" are expressible in $\mathscr{L}^{\omega}_{\infty\omega}(C)$, but not in $\mathscr{L}^{\omega}_{\infty\omega}(Q)$, where Q is an arbitrary finite sequence of simple unary generalized quantifiers.*

Since no finite number of simple unary quantifiers can express the Härtig quantifier, it is interesting to consider logics of the form $\mathscr{L}^{\omega}_{\infty\omega}(I, Q)$, where Q is a finite sequence of simple unary quantifiers. For every such sequence Q, the logic $\mathscr{L}^{\omega}_{\infty\omega}(I, Q)$ constitutes a proper extension of $\mathscr{L}^{\omega}_{\infty\omega}(Q)$. In what follows, we delineate the expressive power of the logics $\mathscr{L}^{\omega}_{\infty\omega}(I, Q)$.

In Section 2.5 we introduced the quantifier MORE and pointed out that it can readily define the Härtig quantifier. In turn, MORE is easily expressible in $\mathscr{L}^{\omega}_{\infty\omega}(C)$ (this also follows from Proposition 2.18). Our next result provides an optimal lower bound for the expressibility of MORE and, at the same time, reveals that MORE is stronger than the Härtig quantifier. The proof uses the method developed in proving Theorem 5.3.

Theorem 5.5. *Suppose Q is a finite sequence of simple unary generalized quantifiers. Then the quantifier MORE is not expressible in $\mathscr{L}^{\omega}_{\infty\omega}(I, Q)$.*

Proof. Let σ be the vocabulary consisting of three unary predicates P_1, P_2 and P_3. We construct two structures A and B over σ so that the following hold:

- Player II wins the invariant (k, I, Q)-pebble game on A and B.
- $A \not\equiv_{L^1_{\omega\omega}(\mathrm{MORE})} B$.

Let $n > 8(2k-1)(4^{(2k-1)m} + 2k)$. By applying the pigeon-hole principle to the $(k\ Q)$-coloring $\chi_{k,Q}$, we can choose positive integers a and b of the same color so that $a < b < n/2$ and $|x - y| \geqslant k$ for all distinct x and y in the set $\{a, b, a+b, n-a, n-b, n-a-b\}$. Let $P_1^A = [a]$, $P_2^A = [a+1, a+b]$, $P_3^A = [a+b+1, n]$, $P_1^B = [b]$, $P_2^B = [b+1, a+b]$ and $P_3^B = [a+b+1, n]$.

Since $|P_1^B| > |P_2^B|$, but $|P_1^A| < |P_2^A|$ the sentence MORE$x, y(P_1(x), P_2(y))$ is true in B but false in A. So we only have to prove that II wins the invariant (k, I, Q)-pebble game on A and B. The part of the proof which corresponds to the $(k\ Q)$-pebble game is an easier version of the proof of Theorem 5.3. So we present only the new case which arises from the quantifier I. Let us suppose r pebbles $(r < k)$ have been played and

Player I moves next. He chooses subsets X_1 and X_2 of one of the models, with $|X_1| = |X_2|$. We may assume $X_1 \cap X_2 = \emptyset$ since among finite sets $|X_1| = |X_2|$ if and only if $|X_1 - X_2| = |X_2 - X_1|$. Again X_1 and X_2 are invariant under all automorphisms of the model which fix the pebbled elements. Let the pebbled elements of $\boldsymbol{A}$ be $a_1, \ldots, a_r$ and let the corresponding elements of $\boldsymbol{B}$ be $b_1, \ldots, b_r$. We assume as an induction hypothesis that the mapping $a_i \mapsto b_i$ is a partial isomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. Let $A_0 = \{a_1, \ldots, a_r\}$ and $B_0 = \{b_1, \ldots, b_r\}$. Let us assume the sets X_1 and X_2 are subsets of the domain of $\boldsymbol{A}$. The case that X_1 and X_2 are subsets of the domain of $\boldsymbol{B}$ is entirely similar. Let $X_i' = X_i - A_0$. We have for both X_1 and X_2 all the four cases presented in the proof of Theorem 5.3. This would seem to generate 16 cases altogether. However, because $|X_1| = |X_2|$ and because of our choice of a and b, the sets X_1 and X_2 fall into the same case. In each case Player II chooses sets Y_1 and Y_2. In fact Player II has in each case only one choice for both Y_1 and Y_2. Next Player I puts pebbles on elements c_1, c_2. We have to demonstrate how Player II puts his pebbles on elements d_1, d_2 so that the partial isomorphism-condition of pebbled elements is preserved and additionally, d_i is in X_1 or X_2, according to whether c_i is in Y_1 or Y_2. This, however, is entirely routine. □

Corollary 5.6. *The queries "given two vertices a and b, is the degree of a smaller than the degree of b?" and "given two vertices a and b, is the connected component of a smaller than the connected component of b?" are expressible in* $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{C})$*, but not in* $\mathscr{L}^{\omega}_{\infty\omega}(I, \boldsymbol{Q})$*, where* $\boldsymbol{Q}$ *is an arbitrary finite sequence of simple unary generalized quantifiers.*

We shall now consider the query "E is an equivalence relation with an even number of equivalence classes". We shall first show that this query is expressible in the extension of $\mathscr{L}^{\omega}_{\infty\omega}$ by MORE and another unary generalized quantifier. Let EM, the *even multiple* quantifier, be the quantifier of type (1, 1) that consists of all structures (A, X, Y) such that $X \subseteq A$, $Y \subseteq A$, and $|X|$ is an even multiple of $|Y|$, i.e.

$$EM = \{(A, X, Y)\colon |X| = m|Y| \text{ for some even number } m\}.$$

Proposition 5.7. *The query "E is an equivalence relation with an even number of equivalence classes" is expressible in* $\mathscr{L}^{4}_{\infty\omega}(MORE, EM)$.

Proof. Let

$$\varphi_1(x) \Leftrightarrow \neg \exists y MORE u, v(uEx, vEy),$$

$$\varphi_{n+1}(x) \Leftrightarrow \forall y(MORE u, v(uEx, vEy) \rightarrow \forall x(x = y \rightarrow (\varphi_1(x) \vee \cdots \vee \varphi_n(x))).$$

Now $\varphi_n(x)$ expresses the property of x that the size of its equivalence class is nth in the ascending order of all sizes of equivalence classes. Of course, there may be several classes of the same size. Let

$$\eta(x) \Leftrightarrow EMu, v(Iy, v(yEu, vEx), vEx).$$

This formula says that there are an even number of equivalence classes of the same size as the equivalence class of x. Let

$$\eta_n^0 \Leftrightarrow \forall x(\varphi_n(x) \rightarrow \eta(x)),$$

$$\eta_n^1 \Leftrightarrow \neg \forall x(\varphi_n(x) \rightarrow \eta(x)).$$

Finally, let θ be the disjunction of all sentences

$$\exists x \varphi_n(x) \wedge \neg \exists x \varphi_{n+1}(x) \wedge \bigwedge_{j=1}^{n} \eta_j^{d_j},$$

where n runs through integers >0 and the sequence $d_1, \ldots, d_n$ runs through all possible sequences with $d_i \in \{0, 1\}$ and $d_i = 1$ for an even number of i. Now $\theta \in \mathscr{L}^4_{\infty\omega}(MORE, EM)$ and θ expresses the property of E that there are an even number of equivalence classes. □

This upper bound in terms of two quantifiers of type (1, 1) is quite tight. Indeed, we show next that no finite sequence of simple unary quantifiers can capture this query, even if the Härtig quantifier is also present.

Theorem 5.8. *Suppose* $\boldsymbol{Q}$ *is a finite sequence of simple unary quantifiers. Then the query "is E an equivalence relation with an even number of equivalence classes?" is not expressible in* $\mathscr{L}^{\omega}_{\infty\omega}(I, \boldsymbol{Q})$.

Proof. The proof uses the $(k\ \boldsymbol{Q})$-coloring method. We shall make use of the following known result from Ramsey theory (see [20]).

Theorem 5.9 (Folkman's Theorem). *For all natural numbers k and c there exists a natural number* $F(k, c)$ *so that if the set* $\{1, \ldots, n\}$ *is c-colored, then there are distinct* $c_1, \ldots, c_k$ *so that all sums* $\sum_{i \in K} c_i$, *where* $K \subseteq \{1, \ldots, k\}$, *have the same color.*

Let σ be the vocabulary consisting of one binary predicate symbol E. We construct two structures $\boldsymbol{A}$ and $\boldsymbol{B}$ on σ so that the following hold:

- In both models E is a equivalence relation on the domain of the model.
- E^A has an even number of equivalence classes.
- E^A has an odd number of equivalence classes.
- Player II wins the invariant $(k, I, \boldsymbol{Q})$-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$.

We may assume k is odd. Let $n = 2F(2k^2 4^k, 4^{(2k-1)m} + 2k)$. We let the universe of both $\boldsymbol{A}$ and $\boldsymbol{B}$ be the set $[n]$. By applying Folkman's Theorem to the $(k\ \boldsymbol{Q})$-coloring $\chi_{k,\boldsymbol{Q}}$, we can choose positive integers $c_1, \ldots, c_k$ so that the numbers $\sum_{i \in K} c_i$, $K \subseteq \{1, \ldots, k+1\}$, are all of the same $\chi_{k,\boldsymbol{Q}}$-color. We may additionally require:

(*) $|\Sigma_1 - \Sigma_2| \geqslant k$ and $|n - \Sigma_1 - \Sigma_2| \geqslant k$ for sums Σ_1 and Σ_2 of two disjoint nonvoid sets of the numbers $c_1, \ldots, c_k$.

Let $c_0 = n - c_1 - \cdots - c_k$. We let $\boldsymbol{A}$ have equivalence classes $A_0, \ldots, A_k$, where A_i is of size c_i. We let $\boldsymbol{B}$ have equivalence classes $B_0, B_2, \ldots, B_k$, where B_2 is of size $c_1 + c_2$ and B_i is of size c_i for $i \neq 2$. Now $\boldsymbol{A}$ has an even number of equivalence classes and $\boldsymbol{B}$ an odd number. So we only have to prove that II wins the definable (k, I, Q)-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$. Let us suppose r pebbles ($r < k$) have been played and Player I is to move. Let the pebbled elements of $\boldsymbol{A}$ be $a_1, \ldots, a_r$ (without repetitions) and let the corresponding elements of $\boldsymbol{B}$ be $b_1, \ldots, b_r$. If Player I makes a move of the ordinary k-pebble game, the strategy of Player II is easy to describe. So we assume Player I makes a generalized quantifier move.

Suppose Player I chooses a subset X of one of the models, with $([n], X) \in Q_j$. We use the fact that X is invariant under all automorphisms of the model which fix the pebbled elements. Let us assume X is a subset of the domain of $\boldsymbol{A}$. The case that X is a subset of the domain of $\boldsymbol{B}$ is entirely similar. Let for $0 \leqslant i \leqslant k$.

$$I_i = \{j\colon a_j \in A_i\} \quad \text{and} \quad J_i = \{j\colon b_j \in B_i\}.$$

We assume as an induction hypothesis that the mapping $a_i \mapsto b_i$ is a partial isomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, $I_0 = J_0$ and $I_1 = J_2$.

Let

$$S_0 = \{i \leqslant k\colon X \cap A_i = \{a_j\colon j \in I'_i\} \text{ for some } I'_i \subseteq I_i\},$$

$$S_1 = \{i \leqslant k\colon X \cap A_i = A_i - \{a_j\colon j \in I'_i\} \text{ for some } I'_i \subseteq I_i\}.$$

Since X is closed under automorphisms which fix $a_1, \ldots, a_r$, we have that $[0, k]$ is the disjoint union of the sets S_0 and S_1. Now we know X already:

$$X = \left\{a_j\colon j \in \bigcup_{i \in S_0} I'_i\right\} \cup \bigcup_{i \in S_1} (A_i - \{a_j\colon j \in I'_i\}),$$

and we can calculate the cardinality of $|X|$:

$$|X| = \sum_{i \in S_1} c_i + s, \quad \text{where } s = \sum_{i \in S_0} |I'_i| - \sum_{i \in S_1} |I'_i|.$$

Note that $|s| < k$. Since $a_i \mapsto b_i$ is a partial isomorphism, there is a mapping $\pi\colon \{0, \ldots, k\} \to \{0, 2, \ldots, k\}$ so that for all i, $I_i = J_{\pi i}$, and additionally, $\pi 0 = 0$ and $\pi 1 = 2$. Let

$$Y = \left\{b_j\colon j \in \bigcup_{i \in S_0} J'_{\pi i}\right\} \cup \bigcup_{i \in S_1} (B_{\pi i} - \{b_j\colon j \in J'_{\pi i}\}),$$

where $J'_{\pi i} = I'_i$.

Case 1. $0 \in S_0$: Let $S'_0 = \{1\} \cup \pi[S_0]$, if $1 \in S_0$, and $S'_0 = \pi[S_0]$ otherwise, and $S'_1 = \{1\} \cup \pi[S_1]$, if $1 \in S_1$, and $S'_1 = \pi[S_1]$ otherwise. Now, $|Y| = \sum_{i \in S'_1} c_i + s$. Since

$\sum_{i\in S_1} c_i$ and $\sum_{i\in S'_1} c_i$ have the same color, $([n], Y)\in Q_j$. The set Y is the move of Player II.

Case 2. $0\in S_1$: In this case

$$|X| = n - \sum_{i\in S_0} c_i + s.$$

Now, $|Y| = n - \sum_{i\in S'_0} c_i + s$. Since $\sum_{i\in S_0} c_i$ and $\sum_{i\in S'_0} c_i$ have the same color, $([n], Y)\in Q_j$. The set Y is the move of Player II.

Next Player I puts a pebble on some element b of $[n]$. Now Player II puts his pebble on an element a of $[n]$ in such a way that

1. $a = a_j$ if and only if $b = b_j$.
2. $aE^A a_j$ if and only if $bE^B b_j$.
3. $a\in A_0$ if and only if $b\in B_0$.
4. $a\in A_1$ if and only if $b\in B_2$.
5. $a\in X$ if and only if $b\in Y$.

This choice guarantees that Player II can maintain his strategy and play the game without ever losing. It is clear that Player II can in fact find the required a.

Next we have to describe the strategy of Player II in the case that Player I makes a Härtig-quantifier move, that is, chooses two subsets X_1 and X_2 of, say, A. The assumption is that $|X_1| = |X_2|$ and both X_1 and X_2 are definable from $a_1,\dots,a_r$. By what was said in the proof of Theorem 5.5, we may assume $X_1\cap X_2 = \emptyset$. As above, we can calculate the cardinalities of the sets X_1 and X_2. For this, let

$$S_{10} = \{i \leqslant k\colon X_1\cap A_i = \{a_j\colon j\in I'_{1i}\} \text{ for some } I'_{1i}\subseteq I_i\},$$

$$S_{11} = \{i \leqslant k\colon X_1\cap A_i = A_i - \{a_j\colon j\in I'_{1i}\} \text{ for some } I'_{1i}\subseteq I_i\},$$

$$S_{20} = \{i \leqslant k\colon X_2\cap A_i = \{a_j\colon j\in I'_{2i}\} \text{ for some } I'_{2i}\subseteq I_i\},$$

$$S_{21} = \{i \leqslant k\colon X_2\cap A_i = A_i - \{a_j\colon j\in I'_{2i}\} \text{ for some } I'_{2i}\subseteq I_i\}.$$

Here S_{10}, S_{11} and S_{20}, S_{21} are partitions of $[0,n]$. We have the representations:

$$X_1 = \left\{a_j\colon j\in \bigcup_{i\in S_{10}} I'_{1i}\right\}\cup \bigcup_{i\in S_{11}} (A_i - \{a_j\colon j\in I'_{1i}\}),$$

$$X_2 = \left\{a_j\colon j\in \bigcup_{i\in S_{20}} I'_{2i}\right\}\cup \bigcup_{i\in S_{21}} (A_i - \{a_j\colon j\in I'_{2i}\}).$$

Let

$$s_1 = \sum_{i\in S_{10}} |I'_{1i}| - \sum_{i\in S_{11}} |I'_{1i}|,$$

$$s_2 = \sum_{i\in S_{20}} |I'_{2i}| - \sum_{i\in S_{21}} |I'_{2i}|.$$

One of the following cases occurs:

1. $0 \notin S_{11} \cup S_{21}$, $|X_1| = \sum_{i \in S_{11}} c_i + s_1$ and $|X_2| = \sum_{i \in S_{21}} c_i + s_2$. Since $X_1 \cap X_2 = \emptyset$ and $|X_1| = |X_2|$, assumption (*) implies $S_{11} = S_{21} = \emptyset$. Therefore, $X_1 = \{a_j: j \in \bigcup_{i \in S_{10}} I'_{1j}\}$ and $X_2 = \{a_j: j \in \bigcup_{i \in S_{20}} I'_{2j}\}$.

2. $0 \in S_{11} - S_{21}$, $|X_1| = n - \sum_{i \in S_{10}} c_i + s_1$ and $|X_2| = \sum_{i \in S_{21}} c_i + s_2$. By (*) and $|X_1| = |X_2|$, this is impossible.

3. $0 \in S_{21} - S_{11}$, $|X_1| = \sum_{i \in S_{11}} c_i + s_1$ and $|X_2| = n - \sum_{i \in S_{20}} c_i + s_2$. By (*) and $|X_1| = |X_2|$, this is impossible.

4. $0 \in S_{11} \cap S_{21}$, $|X_1| = n - \sum_{i \in S_{10}} c_i + s_1$ and $|X_2| = n - \sum_{i \in S_{20}} c_i + s_2$. By (*) we have $S_{10} = S_{20} = \emptyset$. This contradicts $X_1 \cap X_2 = \emptyset$.

Only case 1 is possible. Thus there is no difficulty for Player II to choose his sets Y_1 and Y_2 as the corresponding sets of pebbled elements and maintain his strategy.

Remark. With a little extra work one can show the following: If $\boldsymbol{Q}$ is a finite sequence of simple unary quantifiers, then the query "is E an equivalence relation with an even number of equivalence classes?" is not expressible in $\mathscr{L}^{\omega}_{\infty\omega}(I, EM, \boldsymbol{Q})$. It is also possible to replace the query "is E an equivalence relation with an even number of equivalence classes?" by the more general query "is E an equivalence relation the number of equivalence classes of which is in A?", where A is an arbitrary but fixed infinite and coinfinite set of positive integers. For these and similar extensions of the above theorem, see [46].

Corollary 5.10. *The queries "is a given graph a union of an even number of maximal cliques?" and "does a given graph have an even number of connected components?" are expressible in $\mathscr{L}^{\omega}_{\infty\omega}(\boldsymbol{C})$, but not in $\mathscr{L}^{\omega}_{\infty\omega}(I, \boldsymbol{Q})$, where $\boldsymbol{Q}$ is an arbitrary finite sequence of simple unary generalized quantifiers. The query "is the chromatic number of a given graph on even number?" is not expressible in $\mathscr{L}^{\omega}_{\infty\omega}(I, \boldsymbol{Q})$, where $\boldsymbol{Q}$ is an arbitrary finite sequence of simple unary quantifiers.*

Proof. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be the models constructed in the proof of Theorem 5.8. Let the graph G consist of the complement of the equivalence relation $\boldsymbol{B}$, and the graph H similarly of the complement of $\boldsymbol{B}$. Then the chromatic number of G is even but the chromatic number of H is odd. □

6. Abstract finite model theory

In his paper [40] Per Lindström presented a general definition of an abstract extension of first-order logic. The study of such extensions was subsequently called abstract model theory. The idea was to have the various known extensions of first-order logic, such as infinitary languages and logics with generalized quantifiers, as instances of a single concept. Once the concept was there, it was meaningful to pose questions, like the following: Does first-order logic permit a simple characterization among all abstract logics? Lindström answered this question affirmatively in a number of interesting ways, and this really got the subject into a good start.

The 1970s were a period of intense research activity on abstract model theory, but only the beginning of finite model theory. In its present form, abstract model theory does not cover finite model theory and, as a result, the fundamental notions of abstract model theory must be modified and reformulated if they are to apply to logics defined on finite models only. There is a strong motivation for undertaking a study of abstract finite model theory. Indeed, first-order logic on finite structures is so weak that the need of studying its extensions seems more urgent than on infinite structures.

Our purpose in this section is to modify Lindström's [40] definition of an abstract logic in such a way that it covers also logics based on restricted classes of structures. We shall then discuss, how this new concept may help us bring some order into the family of various logics on finite structures.

We use, as much as possible, the notation of [8]. A *vocabulary* is a set of relation, function and constant symbols. A vocabulary may be also many-sorted. We use τ, σ, ... to denote vocabularies. We fix a binary predicate symbol $\prec$ to denote the ordering of finite ordered structures. The following vocabularies will be frequently referred to:

$$\mathscr{T}_s = \{\tau \mid \tau \text{ is an arbitrary single-sorted vocabulary}\},$$

$$\mathscr{T}_\prec = \{\tau \in \mathscr{T}_s \mid \prec \in \tau\}.$$

The *universe* of a many-sorted structure is defined as the union of the various universes of different sorts. We shall consider below subclasses of the class of all structures. This is where our definition of an abstract logic differs from the usual one. We allow the logic to be defined relative to a restricted class of structures, like the class of all finite structures, or the class of all ordered finite structures. Another new feature in our definition is that we declare already in the definition of a logic which variable symbols are allowed. This is because we want to include logics with a limited finite number of variables only. The variable symbols may be for elements of arbitrary sort or for a limited collection of sorts. We use $\mathscr{V}$ to denote a set of variable symbols. Let $Assgn_{A,\mathscr{V}}$ be the class of *assignments* of variables of $\mathscr{V}$ in A, i.e., the class of functions f with $\mathscr{V}$ as domain and elements of the universe of A of the appropriate sort as values.

Definition 6.1. An *abstract logic on* $\mathscr{K}$ is a 5-tuple $L = (\mathscr{L}, \mathscr{T}, \mathscr{K}, \mathscr{V}, \models_L)$, where

(1) $\mathscr{T}$ is a set of vocabularies.

(2) $\mathscr{L}$ and $\mathscr{K}$ are mappings defined on vocabularies $\tau \in \mathscr{T}$ such that $\mathscr{L}[\tau]$ is a class (the class of $\mathscr{L}$*-formulas* of vocabulary τ) and $\mathscr{K}[\tau]$ is a class of τ-structures.

(3) $\mathscr{V}$ is a set of variable symbols.

(4) $\models_L$ is a relation of elements of $\mathscr{K}[\tau]$, $\mathscr{L}[\tau]$ and $Assgn_{A,\mathscr{V}}$, where τ is an arbitrary vocabulary and $A \in \mathscr{K}[\tau]$. If (A, φ, f) is in the relation $\models_L$, we write $A \models_L \varphi[f]$. Moreover, the following properties (5)–(9) are required to hold:

(5) If $\tau \subseteq \sigma$, then $\mathscr{L}[\tau] \subseteq \mathscr{L}[\sigma]$.

(6) If $A \in \mathscr{K}[\tau]$ and $A \models_L \varphi[f]$, then $\varphi \in \mathscr{L}[\tau]$.

(7) *Isomorphism property*. If $A, B \in \mathscr{K}[\tau]$, $\varphi \in \mathscr{L}[\tau]$, $f \in Assgn_{A,\mathscr{V}}$, $A \models_L \varphi[f]$ and $\pi : A \cong B$, then $B \models_L \varphi[\pi f]$.

(8) *Reduct property*. If $A \in \mathscr{K}[\tau']$, $\varphi \in \mathscr{L}[\tau]$, $f \in Assgn_{A,\mathscr{V}}$, $A \models_L \varphi[f]$ and $\tau \subseteq \tau'$, then $A|\tau \models_L \varphi[f]$.

(9) *Renaming property*. Suppose $\rho : \tau \to \tau'$ is a bijection mapping sort symbols to sort symbols, predicate symbols to predicate symbols of the same arity, function symbols to function symbols of the same arity and constants to constants so that sorts that symbols of τ are equipped with correspond via ρ to sorts the respective symbols are equipped with in τ'. If $A \in \mathscr{K}[\tau]$, then let $A^\rho \in \mathscr{K}[\tau']$ be the results of renaming objects of A according to ρ. Then for all $\varphi \in \mathscr{L}[\tau]$ there is $\varphi^\rho \in \mathscr{L}[\tau']$ such that for all $A \in \mathscr{K}[\tau]$ and all $f \in Assgn_{A,\mathscr{V}}$,

$$A \models_L \varphi[f] \Leftrightarrow A^\rho \models_L \varphi^\rho[f].$$

The concept "A is a *model* of φ", where $A \in \mathscr{K}[\tau]$ and $\varphi \in \mathscr{L}[\tau]$, is defined to mean $\forall f \in Assgn_{A,\mathscr{V}} (A \models_L \varphi[f])$, and denoted by $A \models \varphi$.

An *abstract logic on finite structures* is an abstract logic on the class $\mathscr{F}$ of all finite structures.

Example 6.2. *First-order logic* **FO** is $(\mathscr{FO}, \mathscr{T}_s, \mathscr{S}, \mathscr{V}_\infty, \models_{\mathbf{FO}})$, where $\mathscr{V}_\infty$ is an infinite set of variable symbols, $\mathscr{FO}[\tau]$ is the set of ordinary first-order τ-formulas, and $\models_{\mathbf{FO}}$ is the ordinary satisfaction predicate of first-order logic. This is the *single-sorted* version. *First-order logic on finite structures* $\mathbf{FO}/\mathscr{F} = (\mathscr{FO}, \mathscr{T}_s, \mathscr{F}, \mathscr{V}_\infty, \models_{\mathbf{FO}})$ is obtained by substituting $\mathscr{F}$ for $\mathscr{S}$. *First-order logic on ordered finite structures* is $\mathbf{FO}/\mathscr{F}_< = (\mathscr{FO}, \mathscr{T}_<, \mathscr{F}_<, \mathscr{V}_\infty, \models_{\mathbf{FO}})$ is obtained by substituting $\mathscr{F}_<$ for $\mathscr{S}$. *First-order logic with k variables on finite structures is* $\mathbf{FO}_k/\mathscr{F} = (\mathscr{FO}_k, \mathscr{T}_s, \mathscr{F}, \mathscr{V}_k, \models_{\mathbf{FO}})$, where $\mathscr{V}_k$ is a set of k variable symbols. The formulas of this logic are obtained from $\mathbf{FO}[\tau]$ by simply leaving out all formulas which contain variables which are not in $\mathscr{V}_k$.

Examples 6.3. *Second-order logic* **SO** is $(\mathscr{L}^2, \mathscr{T}_s, \mathscr{S}, \mathscr{V}_\infty, \models_{\mathbf{SO}})$, where $\mathscr{T}_s$ and $\mathscr{V}_\infty$ are as above, $\mathscr{L}^2[\tau]$ is the set of ordinary second-order τ-formulas, and $\models_{\mathbf{SO}}$ is the ordinary satisfaction predicate of second-order logic. Second-order logic differs from first-order logic in that it allows quantification over n-ary relations on the universe. *Second-order logic on finite structures* is $\mathbf{SO}/\mathscr{F} = (\mathscr{L}^2, \mathscr{T}_s, \mathscr{F}, \mathscr{V}_\infty, \models_{\mathbf{SO}})$. *Second-order logic on ordered finite structures* is $\mathbf{SO}/\mathscr{F}_< = (\mathscr{L}^2, \mathscr{T}_<, \mathscr{F}_<, \mathscr{V}_\infty, \models_{\mathbf{SO}})$.

Example 6.4. *Σ^1_1-fragment of second-order logic* is $(\Sigma^1_1, \mathscr{T}_s, \mathscr{S}, \mathscr{V}_\infty), \models_{\Sigma^1_1})$, where $\Sigma^1_1[\tau]$ is the set of existential second-order τ-formulas

$$\exists R_1 \ldots \exists R_n \varphi,$$

$\varphi \in \mathscr{L}_{\omega\omega}[\tau \cup \{R_1, \ldots, R_n\}]$, and $\models_{\Sigma^1_1}$ is the natural restriction of $\models_{\mathbf{SO}}$. As above, we get a version for finite $(\Sigma^1_1/\mathscr{F})$ and ordered finite structures $(\Sigma^1_1/\mathscr{F}_<)$ by replacing $\mathscr{S}$ by $\mathscr{F}$ or $\mathscr{F}_<$.

Example 6.5. *Monadic Σ^1_1-fragment of second order logic is* $(\Sigma^1_{1,1}, \mathcal{T}_s, \mathcal{S}, \mathcal{V}_\infty, \models_{\Sigma^1_{1,1}})$, where $\Sigma^1_{1,1}[\tau]$ is the set of monadic existential second-order τ-formulas

$$\exists R_1 \ldots \exists R_n \varphi,$$

where $R_1, \ldots, R_n$ are monadic predicate symbols and $\varphi \in \mathcal{L}_{\omega\omega}[\tau \cup \{R_1, \ldots, R_n\}]$, and $\models_{\Sigma^1_{1,1}}$ is the natural restriction of $\models_{SO}$. As above, we get a version for finite $(\Sigma^1_{1,1}/\mathcal{F})$ and ordered finite structures $(\Sigma^1_{1,1}/\mathcal{F}_<)$ by replacing $\mathcal{S}$ by $\mathcal{F}$ for $\mathcal{F}_<$.

Example 6.6. We have *infinitary logic with k variables* $(\mathcal{L}^k_{\infty\omega},, \mathcal{T}_s, \mathcal{S}, \mathcal{V}_k, \models_{\mathcal{L}^k_{\infty\omega}})$ and its versions $\mathcal{L}^k_{\infty\omega}/\mathcal{F}$ for finite models and $\mathcal{L}^k_{\infty\omega}/\mathcal{F}_<$ for ordered finite models.

Example 6.7. For any collection $\mathbf{Q} = \{Q_i: i \in I\}$ of generalized quantifier on $\mathcal{S}$, we have *infinitary logic with k variables and with generalized quantifiers* $\mathbf{Q}$, namely $\mathcal{L}^k_{\infty\omega}(\mathbf{Q}) = (\mathcal{L}^k_{\infty\omega}(\mathbf{Q}), \mathcal{T}_s, \mathcal{S}, \mathcal{V}_k, \models_{\mathcal{L}^k_{\infty\omega}(\mathbf{Q})})$. There is a version $\mathcal{L}^k_{\infty\omega}(\mathbf{Q})/\mathcal{F}$ for finite models and a version $\mathcal{L}^k_{\infty\omega}(\mathbf{Q})/\mathcal{F}_<$ for ordered finite models. If $\mathcal{V}_k$ is replaced by $\mathcal{V}_\infty$, the logics $\mathcal{L}^\omega_{\infty\omega}(\mathbf{Q})$, $\mathcal{L}^\omega_{\infty\omega}(\mathbf{Q})/\mathcal{F}$ and $\mathcal{L}^\omega_{\infty\omega}(\mathbf{Q})/\mathcal{F}_<$ are obtained.

Example 6.8. *The logic of PTIME-properties.* We let **PTIME** be the abstract logic $(\mathcal{P}, \mathcal{T}_s, \mathcal{F}, \mathcal{V}_\infty, \models_{\mathbf{PTIME}})$, where $\mathcal{P}[\tau]$ is the set of PTIME properties of finite τ-structures and $A \models_{\mathbf{PTIME}} \varphi$ if and only if A has the property φ. The properties in $\mathcal{P}[\tau]$ are assumed to be closed under isomorphisms. The logic **NP** of NP-properties is defined similarly. By restricting to ordered finite models we get $\mathbf{PTIME}_<$ and $\mathbf{NP}_<$.

Example 6.9. *Fixpoint logic* is $\mathbf{FP} = (FP, \mathcal{T}_s, \mathcal{S}, \mathcal{V}_\infty, \models_{\mathbf{FP}})$ and, as above, we have also $\mathbf{FP}/\mathcal{F}$ and $\mathbf{FP}/\mathcal{F}_<$. For any collection $\mathbf{Q} = \{Q_i: i \in I\}$ of monotone simple unary generalized quantifier, we have *fixpoint logic with generalized quantifiers* $\mathbf{Q}$: $\mathbf{FP}(\mathbf{Q}) = (FP(\mathbf{Q}), \mathcal{T}_s, \mathcal{S}, \mathcal{V}_\infty, \models_{\mathcal{L}^k_{\infty\omega}(\mathbf{Q})})$ together with $\mathbf{FP}(\mathbf{Q})/\mathcal{F}$ and $\mathbf{FP}(\mathbf{Q})/\mathcal{F}_<$.

Example 6.10. In his definition of fixpoint logic with counting Immerman uses the following type of a many-sorted structure [9]: let v denote a special sort reserved for a set of integers together with arithmetic operations on it. A *counting vocabulary* is a vocabulary which has no relations, functions or constants of sort v except the arithmetic ones. An intended structure for a counting vocabulary consists of a structure A of cardinality n of some vocabulary τ and a disjoint set $\{1, \ldots, n\}$ as the universe of sort v together with basic arithmetic operations on these numbers. We denote such structures by $(A, [n])$ and call them *counting structures*. We use $\mathcal{T}_c$ to denote the set of counting vocabularies. *Immerman's fixpoint logic with counting* is $\mathbf{FP}_c = (FP_c, \mathcal{T}_c, \mathcal{F}, \mathcal{V}_\infty, \models_{\mathbf{FP}_c})$, where FP_c is the class of fixpoints of formulas of FO_c. FO_c is the extension of first-order logic by all counting quantifiers $\exists i\, x$. The semantics of $\exists i\, x$ is defined as follows: If $(A, [n])$ is a counting structure and i is a term denoting an element of $[n]$, then $(A, [n]) \models \exists i\, x \varphi(x)$ if there are at least i elements a of A with $(A, [n]), a \models \varphi(x)$.

One can also account for such diverse things as recursive model theory and logic programming, by considering the class of recursive structures and the class of Herbrand structures, respectively. In this paper, however, we will be focusing on abstract logics on the class $\mathscr{F}$ of all finite structures.

An abstract logic in the sense of Lindström [40] is an abstract logic as above on the class $\mathscr{S}$ of *all* structures. In essence, our definition differs from Lindström's in that the L-satisfaction relation is restricted to structures in $\mathscr{K}$ and all closure properties are *relativized* appropriately to $\mathscr{K}$.

Most of the above abstract logics satisfy all the usual closure properties of abstract logics. The only known failures are the following: Σ_1^1 is not closed under negation because the class of infinite models over the empty vocabulary is Σ_1^1, but the class of finite models is not. Fagin [16] showed that $\Sigma_{1,1}^1/\mathscr{F}$ is not closed under negation. $\Sigma_{1,1}^1/\mathscr{F}$ is not closed under $\forall$, because reachability for undirected finite graphs is $\Sigma_{1,1}^1$ (see [5]), but connectedness is not ([16]). It is not known whether $\Sigma_1^1/\mathscr{F}$, $\Sigma_1^1/\mathscr{F}_<$, **NP** and $\mathbf{NP}_<$ are closed under negation or not.

A few results of abstract model theory still hold for abstract logics on an arbitrary class $\mathscr{K}$ of structures. This is, for example, the case with the result that if the *Craig Interpolation Theorem* holds for an abstract logic L, then *Beth's Definability Theorem* also holds for L. Let us recall this familiar result in our framework and notation.

Definition 6.11. (1) An abstract logic $(\mathscr{L}, \mathscr{T}, \mathscr{K}, \mathscr{V}, \models_L)$ satisfies the *Craig Interpolation Theorem* if for all $\tau \in \mathscr{T}$ and $\tau' \in \mathscr{T}$ and all sentences $\varphi \in \boldsymbol{L}[\tau]$ and $\varphi' \in \boldsymbol{L}[\tau']$, the following is true: if for every $\tau \cup \tau'$-model $\boldsymbol{A}$ and every f, $\boldsymbol{A} \models_f \varphi$ implies $\boldsymbol{A} \models_f \varphi'$, then there is $\theta \in \boldsymbol{L}[\tau \cap \tau']$ so that for every $\tau \cup \tau'$-model $\boldsymbol{A}$ and every f, $\boldsymbol{A} \models_f \varphi$ implies $\boldsymbol{A} \models_f \theta$ and $\boldsymbol{A} \models_f \theta$ implies $\boldsymbol{A} \models_f \varphi'$.

(2) An abstract logic $(\mathscr{L}, \mathscr{T}, \mathscr{K}, \mathscr{V}, \models_L)$ satisfies the *Beth Definability Theorem* if for all $\tau \cup \{R\} \in \mathscr{T}$ and for all $\varphi \in \mathscr{L}[\tau \cup \{R\}]$, where R is a relation symbol not in τ, the following is true: if every τ-model has exactly one expansion to a $\tau \cup \{R\}$-model of φ, then there is a $\theta(x) \in \boldsymbol{L}[\tau]$ so that $\theta(x)$ defines R in every model of φ.

(3) An abstract logic $(\mathscr{L}, \mathscr{T}, \mathscr{K}, \mathscr{V}, \models_L)$ satisfies the *Robinson Consistency Theorem* if for all $T \subseteq \boldsymbol{L}[\tau_1 \cap \tau_2]$, $T_1 \subseteq \boldsymbol{L}[\tau_1]$ and $T_2 \subseteq \boldsymbol{L}[\tau_2]$, where $\tau_1, \tau_2 \in \mathscr{T}$ and $T \subseteq T_1 \cap T_2$, the following is true: if T is complete (i.e. any two models of T satisfy the same sentences of $\boldsymbol{L}[\tau_1 \cap \tau_2]$), T_1 has a model and T_2 has a model, then $T_1 \cup T_2$ has a model.

The standard proof shows that if $\boldsymbol{L}$ is closed under $\neg$ and $\wedge$ and $\boldsymbol{L}$ satisfies the Craig Interpolation Theorem, then $\boldsymbol{L}$ satisfies the Beth Definability Theorem. It is also well-known that $\mathbf{FO}/\mathscr{F}$ satisfies neither the Craig Interpolation Theorem nor the Beth Definability Theorem [24, 21]. On infinite models the Robinson Consistency Theorem implies the Craig Interpolation Theorem [42, 1.3], but this is not true on finite models, as the following simple result shows.

Proposition 6.12. *If $\boldsymbol{L} = (\mathscr{L}, \mathscr{T}, \mathscr{K}, \mathscr{V}, \models_L)$ is an abstract logic such that* $\mathbf{FO}/\mathscr{F} \leqslant \boldsymbol{L}$, *$\mathscr{K} \subseteq \mathscr{F}$ and $\mathscr{K}$ is closed under reducts and expansions, then $\boldsymbol{L}$ satisfies the Robinson Consistency Theorem.*

Proof. Suppose $T \subseteq \boldsymbol{L}[\tau_1 \cap \tau_2]$ is complete and $T_1 \subseteq \boldsymbol{L}[\tau_1]$ and $T_2 \subseteq \boldsymbol{L}[\tau_2]$, where $\tau_1, \tau_2 \in \mathscr{T}$, such that $T \subseteq T_1 \cap T_2$. Suppose furthermore that T_1 has a model $\boldsymbol{A}_1$ and T_2 has a model $\boldsymbol{A}_2$. Let $\boldsymbol{B}_i$ be the reduct of $\boldsymbol{A}_i$ to the vocabulary $\tau_1 \cap \tau_2$, $i = 1, 2$. Since T is complete, $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ satisfies the same sentences of $\mathbf{FO}[\tau_1 \cap \tau_2]$. Hence $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are isomorphic. Let $\boldsymbol{A}$ be an expansion of $\boldsymbol{A}_1$ to a $\tau_1 \cup \tau_2$-structure such that the reduct of $\boldsymbol{A}$ to the vocabulary τ_2 is isomorphic with $\boldsymbol{A}_2$. Now $\boldsymbol{A} \models T_1 \cup T_2$. □

Corollary 6.13. $\mathbf{FO}/\mathscr{F}$ *satisfies the Robinson consistency theorem. As a result, the Robinson consistency theorem does not imply the Craig interpolation theorem on finite structures.*

Thus, some of the basic results of traditional abstract model theory carry over to abstract logics on an arbitrary class $\mathscr{K}$ of structures, but not all. This is particularly true when we consider the class $\mathscr{F}$ of all finite structures. Therefore, it seems that the whole theory has to be redeveloped to cover, for example, the case of finite structures.

The first important results in abstract model theory were the characterizations of first-order logic due to Lindström. Lindström characterized first-order logic as a maximal logic which satisfies the compactness theorem and the downward Löwenheim–Skolem theorem. It is well-known that $\mathbf{FO}/\mathscr{F}$ does not satisfy the compactness theorem. Therefore this characterization is not valid on finite models. In another result, Lindström characterizes first-order logic as a maximal logic which satisfies both the downward and the upward Löwenheim–Skolem theorem. This also fails on finite models as $\mathbf{FO}/\mathscr{F}$ trivially fails to satisfy the upward Löwenheim–Skolem theorem.

Kolaitis and Vardi [35] characterized when a class $\mathscr{K}$ of finite structures is definable by a sentence of $\mathscr{L}^k_{\infty\omega}$. From this, we can obtain a Lindström-type result about the infinitary logic $\mathscr{L}^k_{\infty\omega}$ on finite structures. We prove this result for logics of the form $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$. Before we can state this result, we need a new definition.

Definition 6.14. Let $\boldsymbol{L} = (\mathscr{L}, \mathscr{T}, \mathscr{K}, \mathscr{V}, \models_L)$ be an abstract logic and let $\boldsymbol{Q}$ be a sequence of monotone simple unary quantifiers. We say that $\boldsymbol{L}$ has the *$(k, \boldsymbol{Q})$-Karp property* if whenever $\boldsymbol{A}$ and $\boldsymbol{B}$ are structures in $\mathscr{K}$ such that Player II wins the $(k, \boldsymbol{Q})$-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$, then $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy the same sentences of $\boldsymbol{L}$.

Proposition 6.15. *Let $\boldsymbol{Q}$ be a sequence of monotone simple unary quantifiers and let k be a positive integer. The infinitary logic $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})/\mathscr{F}$ is the biggest abstract logic $\boldsymbol{L} = (\mathscr{L}, \mathscr{T}, \mathscr{K}, \mathscr{V}, \models_L)$ with $\mathscr{K} = \mathscr{F}$ which has the $(k, \boldsymbol{Q})$-Karp property.*

Proof. The abstract logic $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})/\mathscr{F}$ itself satisfies the $(k, \boldsymbol{Q})$-*Karp property* by Theorems 3.4 and 3.9. Suppose then $\boldsymbol{L} = (\mathscr{L}, \mathscr{T}, \mathscr{F}, \mathscr{V}, \models_L)$ satisfies the $(k, \boldsymbol{Q})$-*Karp property*, and $\varphi \in \mathscr{L}[\tau]$. By Theorem 3.9, a necessary and sufficient condition for a class $\mathscr{K}$ of finite models to be definable in $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$ is that $\mathscr{K}$ satisfy the following condition: if $\boldsymbol{A} \in \mathscr{K}$ Player II wins the $(k, \boldsymbol{Q})$-pebble game on $\boldsymbol{A}$ and $\boldsymbol{B}$, then $\boldsymbol{B} \in \mathscr{K}$. If we let $\mathscr{K}$ be the class of finite τ-models of φ, we get that $\mathscr{K}$ is definable in $\mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})$. Thus $\boldsymbol{L} \leqslant \mathscr{L}^k_{\infty\omega}(\boldsymbol{Q})/\mathscr{F}$. $\square$

The preceding result holds for sequences of monotone Lindström quantifiers as well, once Definition 6.14 is extended to this case. The proof is unaffected by this generalization.

Proposition 6.15 is special to finite structures, since on the class $\mathscr{S}$ of all structures the k-Karp property alone cannot characterize $\mathscr{L}^k_{\infty\omega}$ as a maximal logic. This can be seen as follows: Let WO be the Lindström quantifier $\{(A, R)$: R well-orders $A\}$. It is easy to see that $\mathscr{L}^k_{\infty\omega}(WO)$ satisfies the k-Karp property on any structures. However, if infinite structures are allowed, WO is not definable in $\mathscr{L}_{\infty\omega}$ and therefore $\mathscr{L}^k_{\infty\omega}(WO)$ is not a sublogic of $\mathscr{L}^k_{\infty\omega}$. We conclude by posing the following problem.

Problem. *Characterize fixpoint logic or partial fixpoint on finite structures as a unique logic having certain model-theoretic properties.*

As mentioned in the introduction, fixpoint logic has emerged as an important extension of first-order logic on finite structures. Thus, a Lindström-type characterization of fixpoint logic on $\mathscr{F}$ may provide us with new insights for this logic and explain its rich closure properties.

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