An Exposition of Ramsey's Result in Logic
By William Gasarch

1 Introduction

In Ramsey's celebrated paper [5] (see also, [2],[3],[4]) his goal was to solve a problem in logic. In this note we discuss what he proved in logic.

We will first state and prove his theorem in logic for undirected graphs (no self loops), and then we will state and prove his theorem in logic for colored hypergraphs.

Def 1.1

- A *graph* is a pair (V, E) where E is a subset of unordered pairs of distinct elements of V. V is referred to as the set of *vertices*. E is referred to as the set of *edges*.
- A *clique* in a graph is a set of vertices such that every pair of vertices in it has an edge.
- An *independent set* in a graph is a set of vertices such that every pair of vertices in it has an edge.

The following is a subcase of Ramsey's Combinatorial theorem.

Theorem 1.2 *For all m there exists a number $R(m)$ such that, for every graph on $R(m)$ vertices, there is either a clique or independent set of size m.*

Note 1.3 It is well known that $2^{m/2} \le R(m) \le 2^{2m}$. A more sophisticated proof, by David Conlon [1] yields, for all k, $n \ge k^{-D\frac{\log k}{\log\log k}}\binom{2k}{k}$ suffices, where D is some constant. A simple probabilistic argument shows that $n \ge (1 + o(1))\frac{1}{e\sqrt{2}})k2^{k/2}$ is necessary. A more sophisticated argument shown by Spencer [6] (see [3]) shows $n \ge (1 + o(1))\frac{\sqrt{2}}{e}k2^{k/2}$ is necessary.

Def 1.4 A sentence is in *the language of graphs* if it only has the usual logical symbols, E a 2-ary predicate, and $=$. We will interpret such sentences as being about undirected graphs with no self loops. Hence we will implicitly assume (1) $E(x, y)$ iff $E(y, x)$ and (2) $\neg E(x, x)$.

Def 1.5 If ϕ is a sentence in the language of graphs then spec(ϕ) is the set of all n such that there is an undirected graph with no self-loops on n vertices where ϕ is true.

Convention 1.6 For ease of notation we make the following conventions.

- If there is a contiguous string of the same type of quantifiers then all of the variables in it are distinct. Hence
$$(\exists x_1)(\exists x_2)(\forall y_1)(\forall y_2)[\phi(x_1, x_2, y_1)]$$
actually means
$$(\exists x_1)(\exists x_2 \ne x_1)(\forall y_1)(\forall y_2 \ne y_1)[\phi(x_1, x_2, y_1)]$$

- There are no self-loops. Hence $E(x,y)$ means $E(x,y) \wedge x \neq y$.
- E is symmetric. So $E(x,y)$ means $E(x,y) \wedge E(y,x)$.

Example 1.7

1. $$\phi = (\forall x)(\forall y)[E(x,y)].$$

   This states that every pair of distinct vertices has an edge. For all n this is satisfied by K_n. Hence, $\mathrm{spec}(\phi) = \mathsf{N}$.

2. $$\phi = (\exists x,y,z)(\forall w)[E(w,x) \wedge E(w,y) \wedge E(w,z)].$$

   ϕ states that there are three distinct vertices x,y,z such that every $w \notin \{x,y,z\}$ is connected to x and y and z. For all $n \geq 0$ $K_{n,3}$ satisfied ϕ. No graph on 0,1, or 2 vertices satisfies ϕ. Hence, $\mathrm{spec}(\phi) = \{3,4,5,\ldots,\}$. (Note that $K_{0,3}$ satisfies ϕ vacuously.)

3. $$\phi = (\exists x_1)(\exists x_2)(\forall y)[x_1 = y \vee x_2 = y].$$

   ϕ is satisfied by all graphs on 2 vertices; however, it is not satisfied by any other graphs. Hence $spec(\phi) = \{2\}$.

Note that in all three examples $\mathrm{spec}(\phi)$ was either co-finite or finite. We will later see that, for all ϕ, this is the case.

2 Definitions and a Lemma Needed for the Graph case

Lemma 2.1

1. *The following is decidable: Given a sentence ϕ and a graph G, determine if ϕ is true in G.*
2. *The following is decidable: Given a sentence ϕ and a number n, determine if $n \in \mathrm{spec}(\phi)$.*

Proof: Use brute force. ▮

We will use Lemma 2.1 without comment.

3 Ramsey's Theorem in Logic on Graphs

The following is a simple case of what Ramsey proved.

Theorem 3.1 *The following function is computable: Given ϕ, a sentence in the language of graphs of the form*

$$(\exists x_1)\cdots(\exists x_n)(\forall y_1)\cdots(\forall y_m)[\psi(x_1,\ldots,x_n,y_1,\ldots,y_m)]$$

output $\mathrm{spec}(\phi)$. *(*$\mathrm{spec}(\phi)$ *will be a finite or cofinite set; hence it will have an easy description.)*

Proof:
Claim 1: If G satisfies ϕ and $x_1, \ldots, x_n$ are the witnesses then any induced subgraph H of G that contains $x_1, \ldots, x_n$ satisfies ϕ.
Proof of Claim 1:
The statement

$$(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)]$$

is true in H since it is true in G and now there are just less cases to check.
End of Proof of Claim 1

Claim 2:

1. If there exists $N_0 \geq n + 2^n R(m)$ such that $N_0 \in \text{spec}(\phi)$ then

$$\{n+m, n+m+1, \ldots, \} \subseteq \text{spec}(\phi).$$

2. If $n + 2^n R(m) \notin \text{spec}(\phi)$ then

$$\text{spec}(\phi) \subseteq \{0, 1, 2, \ldots, n, n+1, n+2, \ldots, n + 2^n R(m) - 1\}.$$

Proof of Claim 2:
1) Since $N_0 \geq n + 2^n R(m) \in \text{spec}(\phi)$ there exists $G = (V, E)$, a graph on N_0 vertices, where ϕ is true. Let $x_1, \ldots, x_n$ be vertices such that the following is true of G:

$$(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)].$$

Let $X = \{x_1, \ldots, x_n\}$ and $U = V - X$. Note that $|U| \geq 2^n R(m)$. Map every $u \in U$ to $(b_1, \ldots, b_n) \in \{0,1\}^n$ such that

$$b_i = \begin{cases} 0 \text{ if } (u, x_i) \notin E \\ 1 \text{ if } (u, x_i) \in E \end{cases} \tag{1}$$

Hence every $u \in U$ is mapped to a description of how it relates to every element in X. Since $|U| \geq 2^n R(m)$ there exists $R(m)$ vertices that map to the same vector. Apply Ramsey's theorem to these $R(m)$ vertices to obtain $z_1, \ldots, z_m$ such that the following are true.

- Either the z_i's form a clique or the z_i's form an ind. set. We will assume the z_i's form a clique (the other case is similar).
- All of the z_i's map to the same vector. Hence they all look the same to $x_1, \ldots, x_n$.

Let H_0 be the graph restricted to $X \cup \{z_1, \ldots, z_m\}$. By Claim 1.a H_0 satisfied ϕ. For every $p \geq 1$ we form a graph $H_p = (V_p, E_p)$ on $n + m + p$ vertices that satisfies ϕ.

- $V_p = X \cup \{z_1, \ldots, z_m, z_{m+1}, \ldots, z_{m+p}\}$ where $z_{m+1}, \ldots, z_{m+p}$ are new vertices.
- E_p is the union of the following edges.
  - The edges in H_0,

- For all $1 \le i < j \le n+m+p$ put an edge between z_i and z_j. (If $i, j \le m$ then there is already an edge there.)
- Let $(b_1, \ldots, b_n)$ be the vector that all of the elements of $\{z_1, \ldots, z_m\}$ mapped to. For $m+1 \le j \le m+p$, for $1 \le i \le m$ such that $b_i = 1$, put an edge between z_j and x_i.

As far as X is concerned, all of the $z_1, \ldots, z_{m+p}$ look the same. Hence any subset of the $\{z_1, \ldots, z_{m+p}\}$ of size m will look just like $z_1, \ldots, z_m$ as far as both X is concerned and as far as their connectivity to each other. Hence H_p satisfies ϕ. Hence $n+m+p \in \mathrm{spec}(\phi)$.

2) Assume, by way of contradiction, that some $N_0 > n+2^n R(m) \in \mathrm{spec}(\phi)$. Then, by part 1 of this claim, all $N \ge n+m$ are in $\mathrm{spec}(\phi)$. In particular $n+2^n R(m) \in \mathrm{spec}(\phi)$. This is a contradiction.
End of Proof of Claim 2

We can now give an algorithm for this problem:

1. Input ϕ which begins $(\exists x_1)\cdots(\exists x_n)(\forall y_1)\cdots(\forall y_m)$.
2. Determine if $n+2^n R(m) \in \mathrm{spec}(\phi)$.
   (a) If YES then by Claim 2a
   $$\{n+m, n+m+1, \ldots\} \subseteq \mathrm{spec}(\phi).$$
   For $0 \le i \le n+m-1$ test if $i \in \mathrm{spec}(\phi)$. We now know the finite set of numbers that are not in $\mathrm{spec}(\phi)$. Call this set NOT. Output $\mathrm{spec}(\phi)$ *is* $\mathsf{N} - NOT$.. Note that $\mathrm{spec}(\phi)$ is cofinite.
   (b) if NO then, by Claim 2b
   $$\mathrm{spec}(\phi) \subseteq \{n+1, n+2, \ldots, n+2^n R(m)\}.$$
   Determine, for each N in this set, which ones are in $\mathrm{spec}(\phi)$. Output that finite set.

▮

4 The Colored Hypergraph Case

We discuss how to generalize Theorem 3.1. Theorem 3.1 was about a structure with a symmetric non-reflexive binary relation E. We will now look at symmetric non-reflexive $\le a$-ary relations. We will allow i-ary relations for $1 \le i \le a$. By non-reflexive we also mean that $E(x_1, x_1, x_2, x_3)$ is false. We will later discuss what to do in the case of non-symmetric relations that are allowed to be reflexive.

Def 4.1 Let $n, a \in \mathsf{N}$.

1. An (a, c)-hypergraph is the complete hypergraph on n vertices where every edge in $\binom{[n]}{\le a}$ exists and has a color. The colors are from $[c]$.

2. Let G be an (a,c)-hypergraph on $[n]$. Let COL denote the coloring. Let $H \subseteq [n]$. H is *homogenous* if for every $1 \le i \le a$ there exists a color c_i such that for all $A \in \binom{H}{i}$, $COL(A) = c_i$.

The following is a Ramsey's Combinatorial theorem in the form he used it.

Theorem 4.2 *For all a, c, m there exists an number $R_{a,c}(m)$ such that, for every c-coloring of $K_{R_{a,c}}(m)$ there is a homogenous set of size m.*

Proof sketch: Use Ramsey's theorem first on the coloring of a-sets. Obtain a homogenous set H. Then use Ramsey's theorem on the colorings of $a-1$-sets of H. Keep doing this. ▮

We now define terms so we can state and prove a theorem in Logic. Before we looked at the language of graphs. We will look at the language of (a,c)-hypergraphs.

Def 4.3 ϕ is a sentence in the language of (a,c)-hypergraphs if it uses predicates $P_{i,j}$ where $i \in [a]$ and $j \in [c]$. The intended interpretation is that $P_{i,j}(x_1,\ldots,x_i)$ is true iff the edge $\{x_1,\ldots,x_i\}$ is colored j. We will also call ϕ an (a,c)-sentence.

Def 4.4 If ϕ is an (a,c)-sentence then spec(ϕ) is the set of all n such that there is a (a,c)-hypergraph for which ϕ is true.

Convention 4.5 For ease of notation we make the following conventions.

- If there is a contiguous string of the same type of quantifiers then all of the variables in it are distinct. Hence
$$(\exists x_1)(\exists x_2)(\forall y_1)(\forall y_2)[\phi(x_1,x_2,y_1)]$$
actually means
$$(\exists x_1)(\exists x_2 \ne x_1)(\forall y_1)(\forall y_2 \ne y_1)[\phi(x_1,x_2,y_1)]$$
- The hypergraph is strongly non-reflexive. This means that $P_{i,j}(x_1,\ldots,x_i)$ is false if any of the vars are the same. Hence $P_{3,17}(x,y,z$ means $P_{3,17}E(x,y,z) \wedge x \ne y \wedge x \ne z \wedge y \ne z$.
- $P_{i,j}E$ is symmetric. So $P_{i,j}(x,y,z)$ means
$$P_{i,j}(x,y,z) \wedge P_{i,j}(y,x,z) \wedge P_{i,j}(z,x,y) \wedge P_{i,j}(z,y,x) \wedge P_{i,j}(y,x,z) \wedge P_{i,j}(y,z,x)$$
.

Example 4.6

1. $$\phi = (\forall x)(\forall y)(\forall z)[P_{3,1}(x,y,z) \vee P_{3,17}x,y,z].$$
This states that every triple of distinct vertices is either colored 1 or colored 17. For all n this is satisfied by the colored hypergraph that colors every triple 1. Hence, spec$(\phi) = \mathsf{N}$.

2. $$\phi = (\exists x, y, z)(\forall w)[P_{3,4}(x, y, w) \wedge P_{4,7}(x, y, z, w).$$

ϕs states that there are three distinct vertices x, y, z such that every $w \notin \{x, y, z\}$ (x, y, w) is colored 4 and (x, y, z, w) is colored 7. We leave it to the reader to show that $\mathrm{spec}(\phi) = \{3, 4, 5, \ldots, \}$.

Lemma 4.7 *Let $a, c \in \mathsf{N}$.*

1. *The following is decidable: Given an (a, c)-sentence ϕ and an (a, c)-hypergraph G, determine if ϕ is true in G.*
2. *The following is decidable: Given an (a, c)-sentence ϕ and a number n, determine if $n \in \mathrm{spec}(\phi)$.*

Proof: Use brute force. ▮

We will use Lemma 4.7 without comment.

The following is a subcase of what Ramsey proved. Its a subcase since Ramsey allowed non-symmetric and reflexive predicates.

Theorem 4.8 *The following function is computable: Given ϕ, an (a, c)-sentence of the form*

$$(\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)]$$

output $\mathrm{spec}(\phi)$. *(*$\mathrm{spec}(\phi)$ *will be a finite or cofinite set; hence it will have an easy description.)*

Proof:
Claim 1: If G satisfies ϕ and $x_1, \ldots, x_n$ are the witnesses then any induced subgraph H of G that contains $x_1, \ldots, x_n$ satisfies ϕ.
Proof of Claim 1:
The statement

$$(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)]$$

is true in H since it is true in G and now there are just less cases to check.
End of Proof of Claim 1

Claim 2:

1. If there exists $N_0 \geq n + 2^n R_{a,c}(m)$ such that $N_0 \in \mathrm{spec}(\phi)$ then
$$\{n + m, n + m + 1, \ldots, \} \subseteq \mathrm{spec}(\phi).$$
2. If $n + 2^n R_{a,c}(m) \notin \mathrm{spec}(\phi)$ then
$$\mathrm{spec}(\phi) \subseteq \{0, 1, 2, \ldots, n, n + 1, n + 2, \ldots, n + 2^n R_{a,c}(m) - 1\}.$$

Proof of Claim 2:
a) Since $N_0 \geq n+2^n R_{a,c}(m) \in \text{spec}(\phi)$ there exists $G = (V, E)$, an (a, c)-hypergraph on N_0 vertices, where ϕ is true. Let $x_1, \ldots, x_n$ be vertices such that the following is true of G:

$$(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)].$$

Let $X = \{x_1, \ldots, x_n\}$ and $U = V - X$. Note that $|U| \geq 2^n R_{a,c}(m)$. Map every $u \in U$ to $(b_1, \ldots, b_n) \in \{0, 1\}^n$ such that

$$b_i = \begin{cases} 0 \text{ if } (u, x_i) \notin E \\ 1 \text{ if } (u, x_i) \in E \end{cases} \tag{2}$$

Hence every $u \in U$ is mapped to a description of how it relates to every element in X. Since $|U| \geq 2^n R_{a,c}(m)$ there exists $R_{a,c}(m)$ vertices that map to the same vector. Apply Ramsey's theorem to these $R_{a,c}(m)$ vertices to obtain $z_1, \ldots, z_m$ such that the following are true.

For every $1 \leq i \leq a$ there exists a color c_i such that every $A \in \binom{z_1,\ldots,z_m}{i}$ is colored c_i.

Let H_0 be the (a, c)-hypergraph restricted to $X \cup \{z_1, \ldots, z_m\}$. By Claim 1.a H_0 satisfied ϕ. For every $p \geq 1$ we form an (a, c)-hypergraph $H_p = (V_p, E_p)$ on $n + m + p$ vertices that satisfies ϕ.

- $V_p = X \cup \{z_1, \ldots, z_m, z_{m+1}, \ldots, z_{m+p}\}$ where $z_{m+1}, \ldots, z_{m+p}$ are new vertices.
- E_p is the union of the following edges.
  - The edges in H_0, colored as they were in H_0.
  - For all $1 \leq ij \leq a$ color every subset of $z_1, \ldots, z_{m+p}$ of size i the color c_i. (If you are only dealing with a subset of $z_1, \ldots, z_m$ then it will already be colored c_i.)
  - Let $(b_1, \ldots, b_n)$ be the vector that all of the elements of $\{z_1, \ldots, z_m\}$ mapped to. For $m + 1 \leq j \leq m + p$, for $1 \leq i \leq m$ such that $b_i = 1$, put an edge between z_j and x_i.

As far as X is concerned, all of the $z_1, \ldots, z_{m+p}$ look the same. Hence any subset of the $\{z_1, \ldots, z_{m+p}\}$ of size m will look just like $z_1, \ldots, z_m$ as far as both X is concerned and as far as their connectivity to each other. Hence H_p satisfies ϕ. Hence $n + m + p \in \text{spec}(\phi)$.

b) Assume, by way of contradiction, that some $N_0 > n + 2^n R_{a,c}(m) \in \text{spec}(\phi)$. Then, by part 1 of this claim, all $N \geq n + m$ are in $\text{spec}(\phi)$. In particular $n + 2^n R_{a,c}(m) \in \text{spec}(\phi)$. This is a contradiction.

End of Proof of Claim 2

We can now give an algorithm for this problem:

1. Input ϕ which begins $(\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)$.
2. Determine if $n + 2^n R_{a,c}(m) \in \text{spec}(\phi)$.
   (a) If YES then by Claim 2a

   $$\{n + m, n + m + 1, \ldots\} \subseteq \text{spec}(\phi).$$

   For $0 \leq i \leq n + m - 1$ test if $i \in \text{spec}(\phi)$. We now know the finite set of numbers that are not in $\text{spec}(\phi)$. Call this set NOT. Output $\text{spec}(\phi)$ *is* $\mathsf{N} - NOT$.. Note that $\text{spec}(\phi)$ is cofinite.

(b) if NO then, by Claim 2b

$$\mathrm{spec}(\phi) \subseteq \{n+1, n+2, \ldots, n+2^n R(m)\}.$$

Determine, for each N in this set, which ones are in $\mathrm{spec}(\phi)$. Output that finite set.

$\blacksquare$

5 The Full Theorem

Lets say we want to allow things like $P_{i,j}(x,y,z) \neq P_{i,j}(x,z,y)$. Or things like $P_{i,j}(x,y,x)$. What can we do? We will sketch how to reduce this to the case in Theorem 4.8. We can assume that the vertex set is $[n]$.

We can transform a (a,c)-hypergraph which allows asymmetry and repeated values to an (a,C)-hypergraph graph which does not for some $C > c$.

We do an example. Let COL be the coloring of $\binom{[n]}{\le a}$. Assume $a \ge 3$. In the new (a,C)-hypergraph we will color $(1,2,3)$ by the sequence

$COL(1,1,1), (COL(1,1,2), COL(1,1,3), COL(1,2,1), COL(1,2,2), COL(1,2,3), COL(1,3,1), (COL(1,3,$
$COL(2,1,1), (COL(2,1,2), COL(2,1,3), COL(2,2,1), COL(2,2,2), COL(2,2,3), COL(2,3,1), (COL(2,3,$
$COL(3,1,1), (COL(3,1,2), COL(3,1,3), COL(3,2,1), COL(3,2,2), COL(3,2,3), COL(3,3,1), (COL(3,3,$

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