Sane Bounds on Van der Waerden-Type Numbers

Nils Molina Anand Oza Rohan Puttagunta

August 31, 2009

Montgomery Blair High School
Maryland, United States of America

under the guidance of
Professor William Gasarch
University of Maryland, College Park
Maryland, United States of America

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Abstract

In Ramsey theory (a branch of combinatorics) one often proves that a function exists by giving enormous bounds on it. The function's actual values may be much smaller; however, this is not known. In this paper we explore *variants* and special cases of these functions where we obtain much smaller bounds.

The following is known: no matter how the lattice points of a coordinate plane are colored red and blue, there exists a square whose corners are the same color (a *monochromatic* square). In fact, using more than just two colors will still guarantee a monochromatic square (one whose vertices are the same color). So for all c (the number of colors), there is a number $G(c)$ where all colorings with c colors of the lattice points of a $G(c) \times G(c)$ grid will contain a monochromatic square. Unfortunately, the necessary number of points is unknown, but bounds are known. These bounds are enormous (roughly $G(c) \leq 2^{2^c} \times 2^{2^{2^{2^{2c}}}}$), however. So we look at a variant of this problem in order to work with more manageable bounds. Looking for rectangles or right triangles rather than squares makes these bounds polynomial. We explore which grids contain a specific number of rectangles or right triangles.

It is known that if the naturals are colored with two colors, there will be two of the same color that are a square apart. Once again, this is true for any number of colors. It is also true for any polynomial whose constant term is 0 and for multiple polynomials. This is known as the Polynomial Van der Waerden Theorem, or PolyVDW. The bounds on the number of naturals that have to be colored grow very quickly. However, for quadratic polynomials and 2 or 3 colors, there are some cases where the theorem is false, and others where it is true with polynomial bounds. Since PolyVDW does not hold in its entirety for all such polynomials, we try to classify for which of these polynomials and for which numbers of colors the theorem holds and try to find bounds for them.

We explore monochromatic rectangles and right triangles rather than squares in order to get reasonable bounds on the grid sizes necessary to guarantee the existence of these monochromatic shapes. We also explore the necessary grid sizes for multiple rectangles and right triangles. In fact, we find many exact numbers. In addition, we look at variants of PolyVDW and obtain much better upper bounds for these numbers than are known for the standard PolyVDW numbers.

1 Introduction

In Ramsey theory (a branch of combinatorics) one often proves that a function exists by giving enormous bounds on it. The function's actual values may be much smaller; however, this is not known. In this paper we explore *variants* and special cases of these functions where we obtain much smaller bounds.

A basic result in Ramsey Theory is that if each of the lattice points of a sufficiently large grid is colored with one of c colors, the grid is guaranteed to have a monochromatic square (a square whose vertices are all the same color):

Theorem (Square Theorem)**.** *For any number of colors c, there exists some number $G(c)$ such that all colorings of a $G(c) \times G(c)$ grid have a monochromatic square.*

The Square Theorem is actually a corollary of the Gallai-Witt Theorem, which is a multi-dimensional version of Van der Waerden's Theorem (VDW) [11]. VDW relates to coloring the natural numbers. If every number is red or blue, it is always possible to find evenly spaced integers of the same color; that is, a monochromatic arithmetic sequence. This is true for any number of colors and any length arithmetic sequence [15]. VDW is a major result in Ramsey Theory. To read more on Gallai-Witt or VDW see [6, 8, 10]. Before stating VDW formally, let us define some terms:

Definition. $[m]$ is shorthand for $\{1, 2, 3, \ldots, m\}$. If $[c]$ is used for a set of colors, usually all that matters is that it's a set of size c, and any other set of size c may be substituted with the necessary transformations.

Definition. A c-coloring is some assignment of colors using c colors. Formally, we define a c-coloring of a set S as a function $\mathrm{COL} : S \to C$, where C is a set of colors and $|C| = c$ (usually $C = [c]$ for convenience).

Theorem (Van der Waerden's Theorem)**.** *For any $c, k \in \mathbb{N}$ there exists a W such that for all c-colorings of $[W]$, there exists a monochromatic arithmetic sequence of length k.*

However, W becomes very large very quickly; in fact, only six of these numbers are known [6]:

c	k	W
2	3	9
3	3	27
4	3	76
2	4	35
2	5	178
2	6	1132

Gowers proved the best upper bound for W, $2^{2^{c^{2^{2^{k+9}}}}}$ [7]. Although Gowers's bound is much better than the original bound ([15] but also see [6, 8]) and Shelah's primitive recursive bound ([14] but also see [8]), it may still be far larger than the actual value of $W(k, c)$.

So we also explore variants on VDW such as PolyVDW. PolyVDW, the Polynomial Van der Waerden Theorem, is a sweeping generalization of ordinary VDW. One could consider VDW as a set of linear polynomials encoding offsets. We can restate the original VDW Theorem using linear polynomials instead of arithmetic sequences.

Definition. $\mathrm{COL}(n)$ represents the color of $n \in \mathbb{N}$ for some coloring.

Theorem (Van der Waerden's Theorem)**.** *For any $k \in \mathbb{N}$, let P be the set of (linear) polynomials $\{x, 2x, \ldots, (k-1)x\}$. For any $c \in \mathbb{N}$, there exists a $W \in \mathbb{N}$ such that for all c-colorings of $[W]$, there exist $a, d \in \mathbb{N}^+$ such that for every polynomial p in P, $\mathrm{COL}(a) = \mathrm{COL}(a + p(d))$.*

Incredibly, this can be extended to any finite set of polynomials P, as long as the constant term of each polynomial is zero. This is the Polynomial Van der Waerden Theorem. More formally,

Theorem (Polynomial Van der Waerden Theorem)**.** *Let P be some finite set of polynomials such that for all polynomial $p(x) \in P$, $p(0) = 0$. For any $c \in \mathbb{N}$, there exists a $W \in \mathbb{N}$ such that for any c-coloring of $[W]$ there exist $a, d \in \mathbb{N}$ such that $\mathrm{COL}(a) = \mathrm{COL}(a + p(d))$ for all $p(x) \in P$.*

PolyVDW was proved for sets containing only one polynomial by Furstenberg [4] and (independently) Sarkozy [12]. The original proof of the full theorem by Bergelson and Leibman [2] uses ergodic methods which give *no* bounds. A later proof by Walters [16] uses purely combinatorial techniques and yields extremely large bounds. (See [6] for an exposition of Walters' proof.) In 2002, Shelah [13] proved a primitive recursive bound for W, but the bound is several levels higher on the Grzegorczyk hierarchy [1] than iterated exponentiation.

Since the bounds are huge, we explore sets with one quadratic where the quadratic has a constant term. We characterize exactly for which quadratics the naturals can be 2-colored while ensuring $\mathrm{COL}(a) \neq \mathrm{COL}(a + p(d))$. We also characterize exactly which quadratics of the form $x^2 + a$ can be 3-colored in this manner. We make progress on the 4 color version, and present a few theorems about the general c color case.

The Square Theorem, stated earlier, states that a large enough grid must contain a monochromatic square when colored with c colors. The best known bound on the minimal grid size $G(c) \leq 2^{2^{O(c)}} \times 2^{2^{2^{2^{2^{O(c)}}}}}$ may be far larger than the actual minimal grid size [6, 8].

We explore variants that lead to lower bounds. One such variant is looking at the minimal grid sizes for monochromatic rectangles. We find *polynomial* bounds on the grid size for multiple monochromatic rectangles. We also find exact numbers for 2 colors and up to 6 rectangles, as well as 3 colors and 2 rectangles. We also find exact numbers for 2 colors and up to 3 monochromatic rectangles of the same color, all of which are quite small. In addition, we make progress on characterizing exactly which grids must contain one rectangle for 4 colors.

Another variant is to look for right triangles rather than rectangles. A right triangle is an even more basic pattern than a rectangle, and indeed we find lower minimal grid sizes to guarantee this shape: the bounds are *linear* in c. We also find how often right triangles must occur in a large c-colored $n \times n$ grid.

2 PolyVDW with Constants

Since the known upper bounds on W for the Polynomial Van der Waerden Theorem are enormous, we explore sets consisting of a single quadratic with a non-zero constant term to find more manageable bounds for W, although the theorem does not hold for all such sets. Sets consisting of one quadratic (with a zero constant term) have been analyzed before [5]. That paper looked at finding bounds for those sets. We attempt to discover how the Polynomial VDW Theorem can be extended to sets consisting of one quadratic with a non-zero constant term.

A weak form of the Polynomial Van der Waerden Theorem states that for any polynomial $p(x) \in \mathbb{Z}[x]$ such that $p(0) = 0$ (that is, p does not have a constant term), and any number of colors c, it is impossible to color every integer such that no two integers are separated by $p(x)$ for some $x \in \mathbb{Z}, x \neq 0$ and are the same color.

Theorem (Weak PolyVDW)**.** *For all $p(x) \in \mathbb{Z}[x] : p(0) = 0, c \in \mathbb{N}$, for any c-coloring $\mathrm{COL} : \mathbb{Z} \to [c]$, there exist $a, d \in \mathbb{N}, d \neq 0$ such that $\mathrm{COL}(a) = \mathrm{COL}(a + p(d))$*

This weak version of PolyVDW has reasonable bounds for 2 and 3 colors. We will investigate polynomials such that $p(0) \neq 0$ (that is, they have a nonzero constant term). We hope the bounds will be reasonable.

First, a few definitions:

Definition. A polynomial $p(x)$ is said to be c-colorable if and only if there exists a coloring $\mathrm{COL} : \mathbb{Z} \to [c]$ such that there does *not* exist $n, x \in \mathbb{Z}, x \neq 0$ such that $\mathrm{COL}(n) = \mathrm{COL}(n + p(x))$. Such a coloring is called a valid coloring. Informally, $p(x)$ is c-colorable iff Weak PolyVDW "does not hold."

Definition. A set S is said to be c-colorable under the polynomial $p(x)$ if and only if there exists a coloring $\text{COL} : S \to [c]$ such that there do not exist $n \in S, x \neq 0 \in \mathbb{Z}$ such that $n + p(x) \in S$ and $\text{COL}(n) = \text{COL}(n + p(x))$.

Definition. A *forbidden distance* is an integer d such that in any valid coloring, $\text{COL}(n) \neq \text{COL}(n+d)\ \forall n \in \mathbb{Z}$.

Definition. A *repeat distance* is an integer d such that in any valid coloring, $\text{COL}(n) = \text{COL}(n + d)\ \forall n \in \mathbb{Z}$.

We shall begin by investigating special cases of quadratic polynomials.

The polynomial $p(x) = x^2 + a$

2 colors

Theorem 2.1. *The polynomial $x^2 + a$ is not 2-colorable, for any choice of $a \in \mathbb{Z}$.*

Proof. There is an x such that $x^2 + a$ is odd and positive (select a sufficiently large x such that $x \equiv a \pmod 2$). Then, consider the three numbers $r = \frac{x^2+a-1}{2}$, $s = \frac{x^2+a+1}{2}$, and x. These are all integers, since $x^2 + a$ is odd. Since

$$(r^2 + a) + (x^2 + a) = (s^2 + a),$$

each pair taken from the three integers $\{1, 1 + (r^2 + a), 1 + (s^2 + a)\}$ is separated by a forbidden distance. By the Pigeonhole Principle, two must be the same color and no coloring can be valid. □

We can get very good bounds on this theorem.

Theorem 2.2. *An upper bound on W such that $[W]$ is not 2-colorable under the polynomial $x^2 + a$ is given by $W \leq \frac{1}{4}\left(a^2 + 8a + 4\right)$ when a is nonnegative, and the bound is linear in $|a|$ when $a < 0$.*

Proof. We shall only show the $a \geq 0$ case here, as the other case is similar.

If $a \equiv 0 \pmod 2$, select $x = 1$; if $a \equiv 1 \pmod 2$, select $x = 0$. Either way, $x^2 + a \leq a + 1$. Now, look back to the proof of Theorem 2.1 and note that since

$$q = \frac{x^2 + a + 1}{2} \leq \frac{(a + 1) + 1}{2},$$

we have

$$1 + q^2 + a \leq \frac{1}{4}\left(a^2 + 8a + 8\right) = 1 + \left(\frac{a + 2}{2}\right)^2 + a.$$

Let $W = \frac{1}{4}\left(a^2 + 8a + 4\right)$. Clearly, $\{1, 1 + p^2 + a, 1 + q^2 + a\} \subseteq [W]$, and we already know this set of three elements is not 2-colorable, so neither is $[W]$.

The proof for negative a is essentially the same (but uses $|a|$ instead of a when appropriate), and in fact, we can choose x carefully to get a smaller magnitude for $x^2 + a$: on the order of $\sqrt{|a|}$ (instead of $a + 1$ as in the previous case). This gives us a bound that is quadratic in $\sqrt{|a|}$, so it's linear in $|a|$. □

3 colors

Theorem 2.3. *The polynomial $x^2 + a$ is 3-colorable if and only if $a \equiv 1 \pmod 3$.*

Proof. We will first prove that if $a \equiv 1 \pmod 3$, then $x^2 + a$ is 3-colorable. Consider the coloring

$$\begin{aligned} \text{COL} : \mathbb{Z} &\to \mathbb{Z}_3 \\ n &\mapsto n \bmod 3 \end{aligned}$$

Because $x^2 + 1 \not\equiv 0 \pmod 3$ (because -1 is a quadratic non-residue mod 3), $\text{COL}(n) \neq \text{COL}(n + (x^2 + a))$.

Now, we will prove that if $a \not\equiv 1 \pmod 3$, then $x^2 + a$ is not 3-colorable. By selecting $x \equiv 3(a+1) \pmod 6$, we can construct an odd value of $x^2 + a$. We let $p = x^2 + a$ and $q = \left(\frac{p-1}{2}\right)^2 + a$, and note that these are both forbidden distances. Then, $p + q = \left(\frac{p+1}{2}\right)^2 + a$, another forbidden distance.

Through some simple testing, one can see that in a valid 3-coloring, $p + 2q$ and $3p$ are repeat distances whenever $p, q, p+q$ are all forbidden distances. This clearly implies that for any y, z, $y(p+2q) + z(3p)$ is a repeat distance. Since the Diophantine equation $sy + tz = \gcd(s,t)$ has a solution (y, z), we know that $\gcd(p + 2q, 3p)$ is a repeat distance as well.

Simple substitution and algebra gives

$$p + 2q = \frac{p^2 + 1}{2} + 2a.$$

Considering this modulo 3, and recalling that $x \equiv 0 \pmod 3$, we have

$$\begin{aligned} p + 2q &\equiv 2(p^2 + 1) + 2a && \pmod 3 \\ &\equiv 2(x^4 + 2ax^2 + a^2 + 1) + 2a && \pmod 3 \\ &\equiv 2(a^2 + 1) + 2a && \pmod 3 \\ &\equiv 2(a^2 + a + 1) && \pmod 3 \end{aligned}$$

Now, simply substitute in the two possible values of $a \pmod 3$.

$$\begin{aligned} a \equiv 0 \quad &\longrightarrow \quad p + 2q \equiv 2 \pmod 3 \\ a \equiv 2 \quad &\longrightarrow \quad p + 2q \equiv 2 \pmod 3 \end{aligned}$$

Let $g = \gcd(p + 2q, 3p)$. From the above congruences, it is clear that $3 \nmid (p + 2q)$. Because $g \mid (p + 2q)$, we know $3 \nmid g$, or equivalently, since 3 is prime, $\gcd(3, g) = 1$. Because $g \mid 3p$ and $\gcd(g, 3) = 1$, we can conclude that $g \mid p$. Since p is a forbidden distance, and any multiple of g is a repeat distance, it is clear that there is no valid 3-coloring, since $\mathrm{COL}(n)$ and $\mathrm{COL}(n + p)$ would have to be both equal and unequal, a contradiction. □

4 colors

We have proved many classes of polynomials to be 4-colorable by constructing explicit colorings. However, it remains an open question to further classify values of a such that $x^2 + a$ is 4-colorable.

Theorem 2.4. *If* $(a \bmod 24) \notin \{0, 8, 12, 15, 20, 23\}$, *then* $x^2 + a$ *is* 4*-colorable.*

Proof. First, note that the condition is equivalent to

$$(a \bmod 24) \in \{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 16, 17, 18, 19, 21, 22\}.$$

Next, notice that the following cases cover all 18 of these congruence classes modulo 24.

Case	Coloring
$a \equiv 1 \pmod 3$	Repeat the block of three (R, G, B) to color all integers using the colors $\{R, G, B, Y\}$. By taking $x^2 + a \bmod 3$, we can verify this coloring.
$a \equiv 1 \pmod 4$	Repeat the block of four (R, G, B, Y) to color all integers using the colors $\{R, G, B, Y\}$. By taking $x^2 + a \bmod 4$, we can verify this coloring.
$a \equiv 2 \pmod 4$	Repeat the block of four (R, G, B, Y) to color all integers using the colors $\{R, G, B, Y\}$. By taking $x^2 + a \bmod 4$, we can verify this coloring.
$a \equiv 3 \pmod 8$	Repeat the block of eight (R, G, R, G, B, Y, B, Y) to color all integers using the colors $\{R, G, B, Y\}$. By taking $x^2 + a \bmod 8$, we can verify this coloring.

□

c colors

We prove general results about c colors, beginning with this one:

Theorem 2.5. *If $-a$ is a square, then $x^2 + a$ is not c-colorable for any finite c.*

Proof. There exists a $b \in \mathbb{Z}$ such that $a = -b^2$. Then, $p(x) = x^2 - b^2 = (x-b)^2 + 2b(x-b)$. This is a polynomial in $x-b$ with no constant term, so we simply apply the Polynomial Van der Waerden Theorem. $\square$

Now, the logical inverse of the previous theorem:

Theorem 2.6. *If $-a$ is not a square, then there is a finite c such that $x^2 + a$ is c-colorable.*

Proof. It is a well-known fact of number theory (proven as Lemma 2.1 below) that if some integer t is not a perfect square, then there exists a prime p such that t is a quadratic non-residue mod p. Now, consider the p-coloring

$$\begin{aligned} \mathrm{COL} : \mathbb{Z} &\to \mathbb{Z}_p \\ n &\mapsto n \bmod p \end{aligned}$$

Evidently $x^2 + a \not\equiv 0 \pmod p$ (since $-a$ is a non-residue mod p), so $\mathrm{COL}(n) \neq \mathrm{COL}(n + (x^2 + a))$. Thus, $x^2 + a$ is p-colorable. $\square$

Lemma 2.1. *For any $t \in \mathbb{N}$, if t is not a square then there exists some prime p where t is a quadratic non-residue mod p.*

Proof. Consider the function $F(x) = x^2 - t$. Two roots are $\pm\sqrt{t}$ (simply plug them in and verify that F is zero), and these must be the only two roots because $F(x)$ has degree two. If t is not a square, these two roots are not rational (either irrational reals or imaginary numbers). Then by [**?**], there exists some prime p such that $F(x) \equiv 0 \pmod p$ has fewer than $\deg F = 2$ solutions, so either no solutions or one solution. Suppose $F(x) \equiv 0 \pmod p$ has 1 solution, $x \equiv r$. Then $-r$ is also a solution, so F has 2 roots $\pmod p$; contradiction. Therefore, $F(x) = x^2 - t$ has no roots $\pmod p$ and a is a quadratic non-residue mod p by definition. $\square$

The polynomial $p(x) = x^2 + ax + b$

We begin by noting that if a is divisible by 2, then we may complete the square and obtain

$$p(x) = \left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 + b,$$

a polynomial in $x = (x + a/2)$ of the form $x^2 + a$ (with different x and a), which may be analyzed using methods covered in the previous section.

2 colors

We have a simple test to find out whether $p(x) = x^2 + ax + b$ is 2-colorable, detailed below.

Theorem 2.7. *The polynomial $p(x) = x^2 + ax + b$ is 2-colorable if and only if a and b are both odd.*

Proof. First, we will prove that if a and b are odd, then $x^2 + ax + b$ is 2-colorable through an explicit coloring. Consider the coloring

$$\begin{aligned} \mathrm{COL} : \mathbb{Z} &\to \mathbb{Z}_2 \\ n &\mapsto n \bmod 2 \end{aligned}$$

Now, it remains to prove that $x^2 + ax + b$ is always odd. We can do this simply by considering it mod 2, using the fact that $a \equiv b \equiv 1 \pmod 2$.

$$\begin{aligned} x^2 + ax + b &\equiv x^2 + x + 1 && \pmod 2 \\ &\equiv x(x+1) + 1 && \pmod 2 \\ &\equiv 1 && \pmod 2 \end{aligned}$$

Now, we will prove that if at least one of a and b is even, then $x^2 + ax + b$ is not 2-colorable. Suppose a is even. Then, as shown earlier, we may transform our polynomial into one of the form $x^2 + a$, which is never 2-colorable by Theorem 2.1. Now, the only remaining case is that a is odd and b is even. Let's take the first difference of $p(x)$.

$$\begin{aligned} p(x+1) - p(x) &= [(x+1)^2 + a(x+1) + b] - [x^2 + ax + b] \\ &= [(2x+1) + (ax + a)] - ax \\ &= 2x + (a+1) \end{aligned}$$

Since $a + 1$ is even (because a is odd), we can conclude that $2x + a + 1$ generates all even integers and therefore $p(x+1) - p(x) = k$ has an integer solution for any even value of k. Now, simply pick a value of y. Note that $p(y) = y(y+a) + b$ is even. Therefore, there exists an x such that $p(x+1) - p(x) = p(y)$. Therefore, the three integers $\{0, p(x), p(x) + p(y)\}$ are all separated by forbidden distances, so it's impossible to 2-color them. □

3 Monochromatic Rectangles

As stated earlier, the Square Theorem guarantees that there exists some $G(c)$ such that for any number of colors c, a *c-coloring* of a $G(c) \times G(c)$ grid must contain a square whose vertices are all the same. However, this $G(c)$ is very large for even small values of c, and the upper bounds become much worse than the VDW bounds [6, 8]. But looking for grids that guarantee monochromatic rectangles instead of squares leads to much smaller grid sizes. In fact, we know that a c-colored $(c^2+c) \times (c^2+c)$ grid must contain a monochromatic rectangle [3], a reasonable bound. However, we can also find exact grid sizes for small number of colors. We know that an extremely large grid colored with c colors will contain a monochromatic rectangle from the Square Theorem. Imagine removing a row or a column one at a time until the grid no longer has to contain a monochromatic rectangle. We can now define some new terms based on this process:

Definition 3.1. A grid G is *c-colorable* if there exists a c-coloring of G such that there are no monochromatic rectangles. So a grid that guarantees a monochromatic rectangle is not c-colorable.

Definition 3.2. A $m \times n$ grid G is *c-minimal* if G is not c-colorable and the grids $(m-1) \times n$ and $m \times (n-1)$ are c-colorable.

Definition 3.3. The Obstruction Set for c colors, denoted OBS_c is the set of c-minimal grids.

The Obstruction Set is useful because a grid contains a monochromatic rectangle if and only if it contains one of the grids in the Obstruction Set. In other words, a *c*-colored $m \times n$ grid contains a monochromatic rectangle if and only if there exists an $a \times b$ grid $G \in OBS_c$ such that $a \leq m$ and $b \leq n$. OBS_2 and OBS_3 are known, shown below [3].
$OBS_2 : \{3 \times 7, 5 \times 5, 7 \times 3\}$
$OBS_3 : \{4 \times 19, 5 \times 16, 7 \times 13, 10 \times 11, 11 \times 10, 13 \times 7, 16 \times 5, 19 \times 4\}$

OBS_4 is not completely known, but is conjectured, based on the following conjecture [3].

Definition 3.4. Some set of points A in a grid G is *rectangle-free* if there are no rectangles that can be formed with these points.

Conjecture (Rectangle Free Conjecture)**.** *For any $m \times n$ grid G, if there exists some rectangle-free set of b points $B \subseteq G$ where $b \geq \lceil \frac{mn}{c} \rceil$, then G is c-colorable.*

Assuming this conjecture, the Obstruction Set is [3]:
$OBS_4 : \{5 \times 41, 6 \times 31, 7 \times 29, 9 \times 25, 10 \times 23, 11 \times 22, 13 \times 21, 17 \times 19, 19 \times 17, 21 \times 13, 22 \times 11, 23 \times 10, 25 \times 9, 29 \times 7, 31 \times 6, 41 \times 5\}$

Coloring 17x17

Though all of our results seem to confirm the Rectangle Free Conjecture, or the RFC for short, attempts at proving it have failed. Thus, OBS_2 and OBS_3 were constructed by finding actual colorings for the various grids. This gets significantly harder for 4 colors, however. By this type of construction, we know that $\{5 \times 41, 6 \times 31, 7 \times 29, 9 \times 25, 25 \times 9, 29 \times 7, 31 \times 6, 41 \times 5\} \subseteq OBS_4$. There are, however, some grids that cannot be resolved using previously established methods. These grids are [3]:
$\{10\times 21, 10\times 22, 11\times 21, 12\times 21, 17\times 17, 17\times 18, 17\times 19, 19\times 17, 18\times 17, 17\times 17, 21\times 12, 21\times 11, 22\times 10, 21\times 10\}$
However, rectangle free subsets of size greater than $\lceil \frac{mn}{c} \rceil$ have been found for most of these grids, which is why assuming the Rectangle Free Conjecture allows us to pin down OBS_4 [3].

One of these grids is 17×17. A rectangle free subset consisting of 74 points has already been found (notice that $\lceil \frac{17*17}{4} \rceil = 73$) [3]. If we assign one color (say Red, or R) to this set of points, we try to find a 3-coloring of the remaining 215 points. Although we did not find such a coloring, we did find multiple rectangle free subsets of size 73 disjoint from the size 74 set already found. These are shown below. If we assign another color (say Blue, or B) to one of these sets, we now have 142 points and 2 colors. So the next exploration would be to try and find a 2-coloring of these 142 points.

```
..R.BB....B.R.RRB
R.....B.RR.R...BB
RBBB.B.R.....RRBR
B..BR..RBBRRR....
RBR.RRB.B.B..B...
.B..BRR.RB..R.B.R
BRB.BR.R.R...B.R.
..BR.R..B.R.BR..B
..R.B...BRRB...BR
RRBR...B.BBBR....
.RR..BRBB..R.RB..
BB.RR.RB.R....R.B
R..B..R...RB.BBRB
B.RR...RR.B.B.BB.
B...RBB.R..B.R.R.
.R.BB.BBR.R.B.R..
...R.B...B.RBB.RR
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..R.B.B....BRBRR.
RB....BBRRBR....B
RBB....R.B..BRR.R
...BR.BR.BRRR.BB.
R.R.RR.B....BBB..
BB.B.RR.R...RB..R
.R.BBR.RBR..B..RB
B..R.R.BB.R..R.B.
B.R..BB..RR.B...R
RRBRB.....B.R.B..
.RR.BBRB.B.R.R...
...RR.R..RBBB.RB.
RB...BR.B.RB..BR.
B.RR...RRB.B....B
..BBR..BR..B.R.R.
.RB..B..R.R..BRBB
...R....BBBR.B.RR
```

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BBR....B....RBRR.
R...B.B.RR.R..BB.
RB..BB.R..B.BRR.R
.B.BR..RB.RRR.B..
R.R.RRB.B.B..B...
..BB.RR.R.BBR...R
.RB.BR.R.R...B.RB
.BBR.RB..BR..R...
B.R..BB..RRB....R
RR.RB..BBB.BR....
BRR...R.B..RBR.B.
.B.RR.R..R.B..RBB
R.....R...RBBBBR.
B.RR...RRBB...B.B
...BR.BBR...BR.RB
.R.B.B..RBR..BRB.
..BR.B.B...R..BRR
```

These grids were found using a computer program, written in Java. It can be found online at `http://www.cs.umd.edu/~gasarch/vdwgang2009/vdwgang2009.html`. The program works by counting the number of rectangles each point in a given set of points is a part of and then removing the point that is in the most rectangles. If there is a tie, it chooses one randomly. It keeps removing points until there are no more rectangles. This process is repeated 100,000 times. Running this program on a 17×17 grid with the rectangle free subset of size 74 removed generated one rectangle free subset of size 73 that is disjoint from the size 74 set per run.

Multiple Rectangles

We also explored what sizes grids need to be to guarantee s rectangles instead of just one rectangle. First we introduce some new notation.

Definition 3.5. A grid G is *c,s-colorable* if there exists a c-coloring of G such that there are fewer than s monochromatic rectangles. So a grid that guarantees s monochromatic rectangles is not c,s-colorable.

Definition 3.6. A $m \times n$ grid G is *c,s-minimal* if G is not c,s-colorable and the grids $(m-1) \times n$ and $m \times (n-1)$ are c,s-colorable.

Definition 3.7. The *ObstructionSet* for c colors, denoted OBS_c^s is the set of c,s-minimal grids.

But before we find the specific Obstruction Sets, we will find some general bounds on these sets.

Bounds

First we identify certain grids as guaranteeing s rectangles.

Lemma 3.1. *If an $n \times m$ grid is not c,s-colorable, an $(n+1) \times m$ grid and a $n \times (m+1)$ can not be $c,s+1$-colorable.*

Proof. Consider the $(n+1) \times m$ grid first. Assume for contradiction that this grid does not have $s+1$ monochromatic rectangles. It must contain s rectangles because it contains an $n \times m$ grid. Remove a row that contains an edge of one of the rectangles. The grid is now once again an $n \times m$ grid, but it only has $s-1$ rectangles. This is a contradiction because an $n \times m$ grid must have at least s monochromatic rectangles. Thus, the $(n+1) \times m$ grid must have $s+1$ monochromatic rectangles. By the same logic the $n \times (m+1)$ grid must have $s+1$ monochromatic rectangles also. □

Theorem 3.1. *If an $n \times m$ grid is not c,s-colorable monochromatic rectangles, an $(n+a) \times (m+b)$ grid can not be $c,s+a+b$-colorable for any a and b.*

Proof. We can think of an $(n+a) \times (m+b)$ grid as one resulting after adding a rows and b columns to the $n \times m$ grid. By Lemma 3.1 each of these additions guarantees one or more monochromatic rectangles. Since we start out with s guaranteed rectangles, the $(n+a) \times (m+b)$ must be guaranteed to have $s+a+b$ monochromatic rectangles. □

Note. This theorem does not mean that a c-colored $(n+a) \times (m+b)$ grid guarantees only $s+a+b$ monochromatic rectangles. It might guarantee more; this theorem establishes a lower bound.

Then we can use the fact that a $(c^2+c) \times (c^2+c)$ grid cannot be c-colored [3] to establish a bound on the Obstruction Set. We also use a Theorem proved in [3] that if an $L \times L$ grid is c-colorable then $|OBS_c| \leq 2(L-c)$. The argument in [3] also proves that if an $L \times L$ grid is s,c-colorable then $|OBS_c^s| \leq 2(L-c)$.

Theorem 3.2.

$$|OBS_c^s| \leq 2c^2 + 2 * \left\lfloor \frac{s}{2} \right\rfloor$$

Proof. We know G_{c^2+c,c^2+c} must have 1 monochromatic rectangle. By Theorem 3.1, a $G_{c^2+c+\lfloor \frac{s}{2} \rfloor, c^2+c+\lfloor \frac{s}{2} \rfloor}$ grid must have $1 + 2 * \lfloor \frac{s}{2} \rfloor$ monochromatic rectangles.

$$\left\lfloor \frac{s}{2} \right\rfloor \geq \frac{s-1}{2}$$

$$1 + 2 * \left\lfloor \frac{s}{2} \right\rfloor \geq 1 + s - 1$$

$$1 + 2 * \left\lfloor \frac{s}{2} \right\rfloor \geq s$$

So we know that a $G_{c^2+c+\lfloor\frac{s}{2}\rfloor, c^2+c+\lfloor\frac{s}{2}\rfloor}$ grid must have at least s monochromatic rectangles. This means that

$$|OBS_c^s| \leq 2(c^2 + c + \left\lfloor \frac{s}{2} \right\rfloor - c)$$

$$|OBS_c^s| \leq 2c^2 + 2 * \left\lfloor \frac{s}{2} \right\rfloor$$

□

Characterizing c,s-colorable Grids

Before we go ahead and find some Obstruction Sets, we develop some theorems that characterize more specifically which grids are c,s-colorable.

Definition 3.8. Placing $x_1, \ldots, x_n \in \mathbb{N}$ in a $n \times m$ grid G for some $n, m \in \mathbb{N}$ is the same as finding a subset of G where the ith row has x_i points for $1 \leq i \leq n$.

Lemma 3.2. *If there are $p + r$ pigeons, p holes, and every pigeon goes into one hole, then there are at least r pairs (not necessarily non-overlapping) of pigeons that must be in the same hole.*

Proof. When $r = 1$, by pigeonhole principle, there must be at least 2 pigeons in 1 hole. So there is at least 1 pair of pigeons in the same hole. Now induct on r, assuming that this holds for r.
If there are $p + r + 1$ pigeons and p holes, then, by pigeonhole principle, there must be at least 2 pigeons in 1 hole. So we have 1 pair of pigeons in the same hole. Remove one of these pigeons to have $p + r$ pigeons in p holes. We know there are at least r pairs of pigeons in the same hole among these pigeons. Since we have one additional pairing, there must be $r + 1$ pairs of pigeons in the same hole. □

Theorem 3.3. *Let $m, n, x_1, \ldots, x_n \in \mathbb{N}$ be such that $(x_1, \ldots, x_n)$ can be placed in a $n \times m$ grid such that there are fewer than s rectangles. Then $\sum_{i=1}^{m} \binom{x_i}{2} \leq \binom{n}{2} + s - 1$.*

Proof. Notice that $\binom{x_i}{2}$ represents the number of pairs of elements such that both of the elements are in row i. So $\sum_{i=1}^{m} \binom{x_i}{2}$ represents the number of pairs of elements such that both elements are in the same row. If a pair does get counted twice, that means that there is a rectangle in this set. Also, $\binom{n}{2}$ represents the total number of pairs of columns.
Now assume for contradiction that it is possible for $\sum_{i=1}^{m} \binom{x_i}{2} > \binom{n}{2} + s - 1$ where the initial conditions are satisfied. Since $\sum_{i=1}^{m} \binom{x_i}{2}$ and $\binom{n}{2} + s - 1$ are integers, we know that $\sum_{i=1}^{m} \binom{x_i}{2} \geq \binom{n}{2} + s - 1$. By Lemma 3.2, considering the pairs of columns as holes and the pairs of elements as pigeons, there must be at least s pairs of pigeons, the pairs of elements, in the same hole, a pair of columns. Each of these s pairs represents a rectangle because they represent a pair of columns where both columns have an element in the same row two different times. So there are s rectangles, which is a contradiction. □

This theorem characterizes the number of rectangles in certain subsets of grids. It is useful when we try to construct a subset of a grid with fewer than s rectangles. The next theorem also characterizes subsets of grids.

Theorem 3.4. *Let $a, n, m \in \boldsymbol{N}$. Let q, r be such that $a = qm + r$ with $0 \leq r < m$. Let G be a $m \times n$ grid. Assume that there exists $A \subseteq G$ such that $|A| = a$ and A does not have s rectangles.*

1. *If $q \geq 2$ then*

$$m \leq \left\lfloor \frac{n(n-1) + 2s - 2rq - 2}{q(q-1)} \right\rfloor$$

2. *If $q = 1$ then*

$$r \leq \frac{n(n-1)}{2} + s - 1$$

Proof. It is proven in [3] that the minimum value of $\sum_{i=1}^{m}\binom{x_i}{2}$ is achieved when there are r rows with $q+1$ points, while the rest have q points. In this case $\sum_{i=1}^{m}\binom{x_i}{2} = (m-r)\binom{q}{2} + r\binom{q+1}{2}$.
So we know that

$$(m-r)\binom{q}{2} + r\binom{q+1}{2} \leq \sum_{i=1}^{m}\binom{x_i}{2} \leq \binom{n}{2} + s - 1$$

by our argument and by Theorem 3.3. Taking the outside terms and simplifying, we get

$$mq(q-1) + 2rq \leq n(n-1) + 2s - 2$$

If $q = 1$

$$2r \leq n(n-1) + 2s - 2$$

$$r \leq \frac{n(n-1)}{2} + s - 1$$

If $q \geq 2$

$$m(q)(q-1) \leq n(n-1) + 2s - 2 - 2rq$$

$$m \leq \left\lfloor \frac{n(n-1) + 2 - 2rq - 2}{q(q-1)} \right\rfloor$$

□

We also investigated which grids have s monochromatic rectangles of the same color. First some more notation:

Definition 3.9. $OBSS_c^s$ signifies the Obstruction set for c colors and s monochromatic rectangles of the same color

Using the pigeonhole principle, it is easy to prove that any grid in $OBS_c^{(cs+1)}$ has s rectangles of the same color, which provides an upper bound for which grids have s monochromatic rectangles of the same color. In other words,

Lemma 3.3. *If a grid is not c,cs+1-colorable, then it must have s monochromatic rectangles of the same color.*

Using this observation, we can also put a bound on $OBSS_c^s$.

Theorem 3.5.

$$|OBSS_c^s| \leq 2c^2 + 2 * \left\lfloor \frac{cs+1}{2} \right\rfloor$$

Proof. We showed in the proof of Theorem 3.2 that $G_{c^2+c+\lfloor \frac{s}{2} \rfloor, c^2+c+\lfloor \frac{s}{2} \rfloor}$ grid must have at least s monochromatic rectangles. So a $G_{c^2+c+\lfloor \frac{cs+1}{2} \rfloor, c^2+c+\lfloor \frac{cs+1}{2} \rfloor}$ must have at least s monochromatic rectangles of the same color by Lemma 3.3. Once again the argument from [3] referenced in the proof of Theorem 3.2 applies and can be used to find a bound on $|OBSS_c^s|$. With some algebra, it is apparent that

$$|OBSS_c^s| \leq 2c^2 + 2 * \left\lfloor \frac{cs+1}{2} \right\rfloor$$

□

It is not true, however, that $OBSS_c^s = OBS_c^{(cs+1)}$. That equality would mean that if a grid has s rectangles of the same color, then it is in $OBS_c^{(cs+1)}$. For example a 2-colored 3×13 grid must have 5 rectangles of the same color, but it is not a part of OBS_2^{11}. There exists a coloring where it has only 8 rectangles.

Now for the Obstruction Sets.

Obstruction Sets

$OBS_2^2 : \{3 \times 8, 4 \times 7, 5 \times 5, 7 \times 4, 8 \times 3\}$
$OBS_2^3 : \{3 \times 9, 4 \times 8, 5 \times 6, 6 \times 5, 8 \times 4, 9 \times 3\}$
$OBS_2^4 : \{3 \times 10, 4 \times 8, 5 \times 6, 6 \times 5, 8 \times 4, 10 \times 3\}$
$OBS_2^5 : \{3 \times 11, 4 \times 9, 5 \times 7, 6 \times 6, 7 \times 5, 9 \times 4, 11 \times 3\}$
$OBS_2^6 : \{3 \times 12, 4 \times 9, 5 \times 7, 6 \times 6, 7 \times 5, 9 \times 4, 12 \times 3\}$
$OBS_3^2 : \{4 \times 20, 5 \times 16, 7 \times 13, 10 \times 11, 11 \times 10, 13 \times 7, 16 \times 5, 20 \times 4\}$
$OBSS_2^2 : \{3 \times 9, 4 \times 8, 5 \times 6, 6 \times 5, 8 \times 4, 9 \times 3\}$
$OBSS_2^3 : \{3 \times 11, 4 \times 9, 5 \times 7, 6 \times 6, 7 \times 5, 9 \times 4, 11 \times 3\}$

The Obstruction Sets require showing that the grids in the sets are c,s-minimal. This entails proving that the grids are not c,s-colorable. All but two of the results proving non-colorability use Theorems 3.1 and 3.4. One of the non-colorability results that could not use those Theorems was proving that a 5×5 grid is not $2, 2$-colorable. This result can be proven through casework. The other one is that a 10×11 grid is not $3, 2$-colorable. This can be proven by using the proof in [3] that proves that the grid is not 3-colorable; the argument also proves that it is not $3, 2$-colorable. However, to prove that that the grids in the Obstruction Sets are c,s-minimal, we also had to show that the all grids smaller than it were c,s-colorable. The colorability results were shown by constructing the actual colorings by hand. The Obstruction Sets for multiple monochromatic rectangles of the same color were derived by constructing all of the necessary colorings. The non-colorability results were derived with the necessary Obstruction Sets for multiple monochromatic rectangles.

4 Monochromatic Right Triangles

Once again, the Square Theorem says a c-coloring of the $a \times b$ integer grid contains a monochromatic square for sufficiently large a and b. The presence of a square implies the existence of a rectangle, which was used in the previous section. Now we shall use the fact that the presence of a square also implies the existence of a right triangle.

Definition 4.1. A *right triangle* is a set of three distinct points $\{(x_1, y_1), (x_1, y_2), (x_2, y_1)\}$, where all the points are the same color. Its *corner* is (x_1, y_1).

General Upper Bounds

In order to find low, reasonable bounds on grid sizes which guarantee a right triangle, we do not use recursive procedures as used for isosceles right triangles by Graham and Solymosi [9], since those give super-exponential bounds. Instead, we classify each point based on its neighbors.

Definition 4.2. A point is a row-point if there is another point of the same color on its row. A point is a nonrow-point if it is not a row-point. Col-points and noncol-points are defined analogously.

Lemma 4.1. *In a coloring without a right triangle, the number of row-points plus the number of col-points cannot exceed the number of points.*

Proof. If there were more row-points and col-points than points, then some point P must be both a row-point and a col-point, making a right triangle with P as its corner. □

Lemma 4.2. *A c-colored row of size $n > c$ points has at least $n - c + 1$ row-points. Similarly a c-colored row of size $m > c$ has at least $m - c + 1$ col-points.*

Proof. There are no more than $c - 1$ nonrow-points per row, or a nonrow-point would be the same color as a row-point. The col-point case is analogous. □

Corollary 4.1. *If $m, n > c$ and $n(m-c+1)+m(n-c+1) > nm$, then a c-colored $n \times m$ grid has a right triangle.*

Theorem 4.1. *For $c \geq 3$ a c-colored $(2c-2) \times (2c-2)$ grid has a right triangle.*

Proof. Suppose a c-coloring of $(2c-2) \times (2c-2)$ with $c \geq 3$ has no right triangle. By Lemma 4.2 each row and column has at least $c-1$ row- and col-points. By Lemma 4.1, this is tight.

Consider a row. Since the row has exactly $c-1$ row-points, it has $c-1$ nonrow-points, all colored differently. The one color that is left must color all $c-1$ row-points. Thus each row has $c-1$ monochromatic row-points and all colors.

By the pigeonhole principle, two rows i and j have the same color of row-points; call it red. Rows i and j have $2c-2$ red row-points between them; if any were on the same column, we'd have a right triangle. In the only other case, we have a red row-point in every column. But since every row contains every color, there must be a red point outside rows i, j, which would form a right triangle with the row-point in its column. ☐

General Lower Bounds

Theorem 4.2. *If an $n \times n$ grid can be c-colored without a right triangle, an $(n+1) \times (n+1)$ grid can be $(c+1)$-colored without a right triangle.*

Proof. Add a row and column to the c-coloring of $n \times n$. Color them with the new color, except for the new corner, which can be any other color. Obviously no right triangles are created. ☐

Theorem 4.3. *For $c \geq 2$, a $\left(\lfloor \frac{3c}{2} \rfloor - 1\right) \times \left(\lfloor \frac{3c}{2} \rfloor - 1\right)$ grid can be c-colored without a right triangle.*

Proof. We show how to c-color $\left(\frac{3c}{2} - 1\right) \times \left(\frac{3c}{2} - 1\right)$ without a right triangle for even c. Then, by Theorem 4.2, this theorem is also true for odd c.

A	A	A	D	E	F	B	C
B	B	B	D	E	F	A	C
C	C	C	D	E	F	A	B
D	E	F	A	A	A	B	C
D	E	F	B	B	B	A	C
D	E	F	C	C	C	A	B
E	D	D	E	D	E	A	B
F	F	E	F	F	F	B	A

As in the example, color the upper left $c \times c$ subgrid using $\frac{c}{2}$ monochromatic row-points on each row and $\frac{c}{2}$ monochromatic col-points on each column, with the first $\frac{c}{2}$ colors used for row-points and the last $\frac{c}{2}$ colors used for col-points. As in the example, the size of the each row can be increased from c to $\frac{3c}{2} - 1$ without creating any row-points using the $\frac{c}{2} - 1$ remaining "row-point colors," and the columns can be similarly extended without creating any col-points. Since neither col- nor row-points were created, the bottom right $\left(\frac{c}{2} - 1\right) \times \left(\frac{c}{2} - 1\right)$ subgrid can be colored by cycling in each row the first $\frac{c}{2} - 1$ colors. ☐

Bounds for Specific c

$OBST_c$ is the Obstruction Set for right triangles with c colors (see Section 3 for a formal definition of Obstruction Set).
$OBST_2 = \{3 \times 3\}$
$OBST_3 = \{4 \times 4\}$
$OBST_4 = \{5 \times 6, 6 \times 5\}$
$OBST_5 = \{6 \times 9, 7 \times 7, 9 \times 6\}$

Corollary 4.1 and Theorems 4.1 and 4.3 can be used to find these results, except in a few cases, which follow.

Theorem 4.4. *All* 4*-colorings of a* 6×5 *grid have a right triangle.*

Proof. Suppose we have a 4-coloring of 6×5 without a right triangle. By the pigeonhole principle, each row has two points of the same color. By the pigeonhole principle, two rows have the same-colored pair. Without loss of generality (since swapping rows and columns does not change the existence of any right triangles), the first two rows both have a pair of color R:

R	R			
		R	R	
*	*	*	*	
*	*	*	*	
*	*	*	*	
*	*	*	*	

In the 4×4 subgrid marked by *, the color R may not be used, or there will be a right triangle, so it must be 3-colored. But all 3-colorings of 4×4 have a right triangle by Theorem 4.1, a contradiction. □

Lemma 4.3. *The* 4*-coloring of a* 5×5 *grid which avoids right triangles derived from Theorem 4.3 is unique, up to swapping of rows and columns, and switching of colors.*

Proof. Consider a 4-coloring of 5×5 without right triangles with the following pattern (A and B are colors):

A	A			
B	B			
(1)	(3)			
(2)	(4)			

In words the pattern is four row-points occupying two columns and two rows, with the added condition that within each row the two row-points be the same color. The above diagram chooses the first two columns and first two rows without loss of generality.

Two elements in the first column are monochromatic by the pigeonhole principle, since there are 3 unknown points with 2 colors possible for each. Without loss of generality we make elements (1) and (2) color C, forcing elements (3) and (4) to be color D. Continuing this kind of logic, the coloring becomes the one from Theorem 4.3.

To finish the proof, we show that there must be a right triangle in a 4-coloring of 5×5 without the previous pattern. Suppose for sake of contradiction there were no right triangles. Consider the most popular color, and call it A. By the pigeonhole principle there are at least $\lceil \frac{5*5}{4} \rceil = 7$ points colored by A. Using the fact that there is no 3-coloring of 4×4, it can be shown that 6 of the As must be arranged in the following way (up to swapping of rows and columns):

A	A			
(1)	(3)		A	(5)
(2)				A
	(4)			A
				A

In the first column there must be by the pigeonhole principle two monochromatic elements. One of the two must be on the second row in order to avoid the previous pattern. The same is true for the second column. Therefore without loss of generality we make (1) and (2) color B and (3) and (4) color C. This forces (5) to be colored with A, creating a right triangle and a contradiction. □

Theorem 4.5. *All* 5*-colorings of* 7×7 *have a right triangle.*

Proof. Suppose for sake of contradiction that we have a 5-coloring of 7×7 without a right triangle. By Lemma 4.1 the number of row-points plus the number of col-points cannot exceed $7 * 7 = 49$. By Lemma 4.2, each row and column has at least 3 row-points or 3 col-points, respectively. This adds up to 42 row- and col-points total. Thus if the total number of rows and columns with more than 3 row- or col-points (respectively) exceeds $49 - 42 = 7$ then there is a right triangle. Consider the 5-coloring of 4×4 given by Theorem 4.3:

A	A	C	D	B
B	B	C	D	A
C	D	A	A	B
C	D	B	B	A
D	C	D	C	A

This coloring has more than 7 rows and columns with "extra" row- or col-points, so it cannot occur in our 5-coloring of 7×7. Since by Lemma 4.3 this is the unique 4-coloring of 5×5 which avoids right triangles, there can be *no* 4-colorings of 5×5 in our colored grid.

Consider the most popular color, and call it R. By the pigeonhole principle, there must be at least $\lceil \frac{7*7}{5} \rceil = 10$ points colored by R. It can be shown by simple casework (using the fact that there can be no 4-colorings of 5×5) that the only way to put 10 R on our 7×7 grid is the following, up to swapping of rows and columns:

R	R	R	R			
				R		
				R		
					R	
					R	
						R
						R

Notice this diagram guarantees that the first row has more than 3 row-points. It also guarantees that the rightmost three columns each have more than 3 col-points, since each of these must have an additional monochromatic pair of points.

Repeatedly applying pigeonhole, the next three most popular colors must have the above configuration (up to swapping of rows and columns). This results in more than 7 "extra" row- or col-points, creating a right triangle and a contradiction. ☐

For the following two thereoms, a computer program was used. The program recursively finds all c-colorings of an $n \times m$ grid without right triangles. At each step it greedily chooses the point with the least number of possible colors and tries each of these colors. To speed up execution greatly, the user can type in partial grids for the program to use as a starting point. The program's source code (in C++) can be found at `http://www.cs.umd.edu/~gasarch/vdwgang2009/vdwgang2009.html`.

Theorem 4.6. *There exists a 5-coloring of 8×6 without a right triangle.*

Proof. Computer search produces:

A	B	C	D	D	E
A	B	C	E	E	D
C	E	D	A	B	C
D	D	C	A	B	E
A	E	D	B	C	B
A	C	D	C	B	E
E	B	E	A	C	D
B	E	B	A	C	D

Interestingly, parts of this coloring (particularly the first three rows) resemble the general c-coloring given in Theorem 4.3. $\square$

Theorem 4.7. *All* 5*-colorings of* 9×6 *have a right triangle.*

Number of Right Triangles

Since right triangles are such a basic structure and thus occur more frequently than other more complicated ones, it is natural to ask, how frequently *do* they occur? Answering this question, Theorem 4.8 describes a lower bound for the number of right triangles on a c-colored $n \times n$ grid. The lower bound is nearly perfect, and occasionally exact, so it gives us a good idea about the density of right triangles in a large $n \times n$ grid.

Lemma 4.4. *For any fixed constants m and v, if $\sum_{i=1}^{v} \frac{1}{x_i} + \frac{1}{y_i} = m$ such that $x_i, y_i \geq 2$ for all i, then $\sum (x_i - 1)(y_i - 1)$ is minimized when all x_i and y_i equal $\frac{2v}{m}$.*

Proof. We need to show that for any sets of x_i and y_i (with $x_i, y_i \geq 2$)

$$\sum_{i=1}^{v} (x_i - 1)(y_i - 1) \geq v \left(\frac{2v}{m} - 1 \right)^2 .$$

It is easy to show using calculus that $(x_i - 1)(y_i - 1)$ is minimized (keeping $\frac{1}{x_i} + \frac{1}{y_i}$ constant) when $x_i = y_i = u_i$ for some u_i. $\sum \frac{2}{u_i} = \sum \frac{1}{x_i} + \sum \frac{1}{y_i} = m$, making the inequality

$$\sum (u_i - 1)^2 \geq v \left(\frac{v}{\sum \frac{1}{u_i}} - 1 \right)^2 .$$

Consider the left side of the inequality. Jensen's inequality [17] says $f(\frac{\sum x_i}{n}) \leq \frac{\sum f(x_i)}{n}$ (where n is the number of x_i) when f is convex, and here we use $f(x) = (x - 1)^2$:

$$\sum (u_i - 1)^2 = v \frac{\sum (u_i - 1)^2}{v} \geq v \left(\frac{\sum u_i}{v} - 1 \right)^2$$

To complete the proof apply Jensen's inequality again, this time with $f(x) = \frac{1}{x}$. $\square$

Lemma 4.5. *Suppose $\sum_{i=1}^{u} \frac{1}{x_i} + \frac{1}{y_i} = k \leq \frac{2u}{3}$ such that $x_i, y_i \in \{1\} \bigcup [2, \infty)$ for all i. Let S be the set of all x_i and y_i such that i satisfies $x_i, y_i \geq 2$. If x_j, y_j is not in S, $\sum (x_i - 1)(y_i - 1)$ can be made smaller, holding k and u constant, by adding x_j, y_j to S, thereby increasing the size of S by one.*

Proof. All (x_i, y_i) pairs not in S contribute at least one to k, but on average an (x_i, y_i) pair contributes no more than $\frac{2}{3}$, since $k \leq \frac{2u}{3}$. Therefore, letting m be the average contribution to k (per pair of numbers) of $S \bigcup \{x_j, y_j\}$ and letting m_o be the average contribution to k (per pair of numbers) of S, $m_o < m \leq \frac{2}{3}$.

Let n be the number of pairs of elements in S. By Lemma 4.4, $S \bigcup \{x_j, y_j\}$ contributes at least $n((\frac{2}{m_o}) - 1)^2$ to $\sum (x_i - 1)(y_i - 1)$. We need to prove that we can make $S \bigcup \{x_j, y_j\}$ contribute less after x_j and y_j are added into S.

To add x_j and y_j into S while keeping k constant, set x_j, y_j and all elements of S to $\frac{2}{m}$. Now we just need to prove that

$$(n + 1)(2/m - 1)^2 < n(2/m_0 - 1)^2$$
$$n \left((2/m_o - 1)^2 - (2/m - 1)^2 \right) > (2/m - 1)^2 .$$

Let $v = \frac{1}{x_j} + \frac{1}{y_j}$ for the previous values of x_j and y_j. Since $k = \sum \frac{1}{x_i} + \sum \frac{1}{y_i}$ is constant, $2n\frac{m}{2} = (2n-2)\frac{m_o}{2} + v$, which simplifies to $m = m_o + \frac{1}{n}(v - m_o)$. Let $z = \frac{1}{m_o} - \frac{1}{m}$. z equals

$$\frac{1}{m - \frac{1}{n}(v - m_o)} - \frac{1}{m} = \frac{\frac{1}{n}(v - m_o)}{m^2 - \frac{m}{n}(v - m_o)}$$

From the first paragraph of this proof, $v > m_o$, $v > 1$ and $m_o < m$, so $z > \frac{1}{n}\frac{1-m}{m^2}$. Substituting z into the left side of the inequality, the right side follows:

$$n\left((2/m - 1 + 2z)^2 - (2/m-1)^2\right) = n\left(4z\,(2/m-1) + 4z^2\right)$$

$$n\left(4z\,(2/m-1) + 4z^2\right) > 4nz\,(2/m-1) > 4\left(\frac{1-m}{m^2}\right)(2/m-1) > (2/m-1)^2$$

Notice that we use the easily verifiable fact $4\frac{1-m}{m^2} > 1$, which requires that $m \leq \frac{2}{3}$. □

Lemma 4.6. *If $\sum_{i=1}^{u} \frac{1}{x_i} + \frac{1}{y_i} = k$ for constants k and u such that $x_i, y_i \in \{1\} \bigcup [2, \inf)$ for all i, then $\sum(x_i - 1)(y_i - 1)$ is minimized when all x_i and y_i are the same.*

Proof. Consider the set S of all x_i and y_i such that i satisfies $x_i, y_i \geq 2$. Suppose there exist numbers x_j and y_j not in S. We can put them in S by changing their values (holding k constant, and making sure everything in S stays in S) so that $\sum(x_i-1)(y_i-1)$ decreases using the operation detailed in Lemma 4.5. If this operation is repeated for each x_i and y_i not in S, we will always get a lower $\sum(x_i-1)(y_i-1)$ than before, and S will eventually contain all x_i and y_i. This establishes the fact that a minimal $\sum(x_i-1)(y_i-1)$ occurs when all x_i and y_i are in S. In other words, a minimal $\sum(x_i-1)(y_i-1)$ occurs when all $x_i, y_i \geq 2$, so by Lemma 4.4, $\sum(x_i-1)(y_i-1)$ is minimized when all x_i and y_i are the same. □

Before we present the theorem, a definition pertaining to a c-colored $n \times n$ grid:

Definition. A row with exactly k elements ($k \geq 0$) of the same color has a k-rowset, which is the set of these k elements. Each row has exactly one rowset for each of the c colors of the grid, meaning an $n \times n$ grid has nc rowsets. These groupings of points on the grid account for each point, so that each point is in exactly one rowset. Similarly, a column with exactly k elements of the same color has a k-colset, which is the set of these k elements.

Theorem 4.8. *A c-coloring of $n \times n$ where $n \geq 3c$ has at least $n^2(\frac{n}{c} - 1)^2$ right triangles.*

Proof. Map each point on $n \times n$ to a different integer i from 1 to n^2. Let x_i be the number of points on i's row (including i itself) which are the same color as i, and let y_i be the number of points on i's column colored with that color. i is the corner of $(x_i-1)(y_i-1)$ right triangles. Since each triangle has exactly one corner, the number of triangles is $\sum(x_i-1)(y_i-1)$.

$\sum \frac{1}{x_i} = nc$, since for each of the nc k-rowsets, you will have k x_i equaling k. (We need not consider the case where $x_i = 0$ because it is never optimal: if a row has zero red points, you can change one of its points to red without creating any right triangles.) The same goes for $\sum \frac{1}{y_i}$, so $\sum \frac{1}{x_i} + \sum \frac{1}{y_i} = 2nc$. Now apply Lemma 4.6 (with $u = n^2$ and $k = 2nc \leq \frac{2u}{3}$ because $n \geq 3c$) to prove $n^2\left(\frac{n}{c} - 1\right)^2$ lower bounds $\sum(x_i-1)(y_i-1)$. □

Note. Since you can evenly distribute colors inside each row and column by cycling colors in each row, this is an exact lower bound on the number of right triangles if c divides n.

Open Questions

We have shown that an even distribution of colors inside each row and column of a large $n \times n$ grid minimizes the number of right triangles. The same seems true for rectangles: in colorings of grids that avoid rectangles found in [3] and in this paper, colors are pretty evenly distributed inside each row and column. This leads us to the following conjecture:

Conjecture. *Let an X-shape be a set of monochromatic points* $\{P_1, P_2, \cdots, P_n\}$ *which satisfies a set of rules particular to X, all of the form "*P_a *is on the same row/column as* P_b*." If you have a sufficiently large* $n \times n$ *grid (where* c *divides* n*), the minimum number of monochromatic X-shapes occurs when colors are distributed evenly in each row and column.*

This conjecture reflects the notion that randomness leads to disorder: it is intuitively most probable for a random grid coloring to have nearly even distributions of colors in each row and column, and the disorder would be a lack of monochromatic shapes.

Additionally, the result about the number of right triangles on an $n \times n$ grid is particularly interesting since it could be directly applied to give bounds for other shapes made up of right triangles.

5 Conclusion

We have shown that by looking at variants of VDW, one can find VDW-type numbers that are polynomial. The structures we explored allowed us to find nice polynomial bounds and many exact figures. Many of the polynomial bounds turned out to be quadratic, a far cry from the towers seen in existing VDW bounds. We did in fact find improved bounds for these variants, in contrast with the enormous bounds for the original theorems. Our hope is that finding small upper bounds, and even exact numbers, for these variants will be a step in the right direction for finding better bounds for VDW, PolyVDW, and Gallai-Witt.

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