Better Bounds on Rado Numbers

Russell Moriarty

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1 Introduction

At present the best known upper bounds on Rado numbers are derived from upper bounds on Van der Waerden (VDW) numbers and hence are large. We present a different proof of Rado's Theorem which leads to better upper bounds on the Rado numbers in some cases.

2 Van der Waerden's Theorem

Recall van der Waerden's theorem:

Theorem 1 (Van der Waerden's Theorem)**.** *For all $k, c \in \mathbb{N}$, there exists $W = W(k,c)$ such that for all c-colorings $\chi : [W] \to [c]$ there exist $a, d \in \mathbb{N}$ such that $\chi(a) = \chi(a+d) = \chi(a+2d) = \cdots = \chi(a+(k-1)d)$.*

This was first proven by van der Warden [6]. See the books by Graham, Rothchild, and Spencer [3], Landman and Robertson [4] or the free on-line book of Gasarch, Kruskal, Parrish [1] for the proof in English. This proof gives enormous upper bounds on the numbers $W(k,c)$ that are not primitive recursive. Shelah [5] gave an alternative proof that yields primitive recursive upper bounds. All of the proofs noted above are elementary. Gowers [2] provided a non-elementary proof that yields much better bounds.

The following variant of Van der Waerden's Theorem is used to prove Rado's theorem. Proofs can be found in the three books given above; however, we do not know its origins. We present a proof for completeness and to contrast the upper bounds it yields to the ones we will obtain later.

Theorem 2 (VDW Variant)**.** *For all $k, l, m, c \in \mathbb{N}$, there exists $U = U(k, l, m, c)$ such that for all c-colorings $\chi : [U] \to [c]$ there exist $a, d \in \mathbb{N}$ such that*

$$\chi(a) = \chi(a + md) = \chi(a + 2md) = \cdots = \chi(a + (k-1)md) = \chi(ld)$$

Proof. The proof is by induction on c. Clearly for all k, l, m, we have that $U(k, l, m, 1) = \max\{1 + (k-1)m, l\}$.

For the induction step we assume $U(k, l, m, c-1)$ exists and use it to prove the existence of $U(k, l, m, c)$. Let χ be a c-coloring of $[W(k', c)]$, where $k' = (k-1)lmU(k, l, m, c-1) + 1$. By the definition of $W(k', c)$, there exist a, d such that

$$\chi(a) = \chi(a + d) = \chi(a + 2d) = \cdots = \chi(a + (k'-1)d)$$

Without loss of generality assume this color is RED. This implies that for all $i \in [U(k, l, m, c-1)]$,

$$\chi(a) = \chi(a + mid) = \chi(a + 2mid) = \cdots = \chi(a + (k-1)mid) = \text{RED}$$

Now there are two cases:

CASE 1: There exists $i \in [U(k, l, m, c-1)]$ such that $\chi(lid) = \text{RED}$. Therefore

$$\chi(a) = \chi(a+mid) = \chi(a+2mid) = \cdots = \chi(a+(k-1)mid) = \chi(lid) = \text{RED}$$

and we are done with this case.

CASE 2: For all $i \in [U(k, l, m, c-1)]$, $\chi(lid) \neq \text{RED}$. This implies that χ gives a $(c-1)$-coloring of $\{ild\}_{i \in [U(k,l,m,c-1)]}$. By the definition of $U(k, l, m, c-1)$, there exist a', d' such that

$$\chi(a'ld) = \chi((a'+md')ld) = \chi((a'+2md')ld) = \cdots = \chi((a'+(k-1)md')ld) = \chi(ld'ld)$$

Substituting $A = a'ld$ and $D = d'ld$ gives

$$\chi(A) = \chi(A + mD) = \chi(A + 2mD) = \cdots = \chi(A + (k-1)mD) = \chi(lD)$$

and we are done. $\square$

The proof of the previous theorem gives the following extremely crude upper bound on $U(k, l, m, c)$.

Corollary 1. *Let $f_{k,l,m,c}(n) = W((k-1)lmn + 1, c)$, and let $f^{(i)}(n)$ denote the composition of f with itself n times. Then:*

$$U(k, l, m, c) \leq f_{k,l,m,c}^{(c-1)}(max\{1 + (k-1)m, l\})$$

Note this upper bound requires $c-1$ recursive calls to Van der Waerden's Theorem!

3 Rado's Theorem

We begin by presenting the lemma used in our proof of Rado's Theorem, then present Rado's Theorem itself.

Lemma 1. *For all $c, l \in \mathbb{N}$ and for all $m \neq 0 \in \mathbb{Z}$, there exists a $P = P(l, m, c) \in \mathbb{N}$ such that for all c-colorings of $[P]$, there exists $a, d \in \mathbb{N}$ such that $a, ld, a + md \in [P]$ are monochromatic.*

Proof. Let $P = U(2, l, m, c)$ and $\chi : [P] \to [c]$ be a coloring of $[P]$. By the definition of U, there exist a, d such that $\chi(a) = \chi(a + md) = \chi(ld)$, which is exactly what we wanted to prove. □

This proves the following crude upper bound on $P(l, m, c)$.

Corollary 2. $P(l, m, c) \leq f^{(c-1)}_{2,l,m,c}(max\{m + 1, l\})$

Theorem 3 (Rado's Theorem)**.** *For all $a_1, \ldots, a_n \in \mathbb{Z}$, if there exists an $I \subseteq [n]$ such that $\sum_{i \in I} a_i = 0$, then for all $c \geq 1$, $\exists R(a_1, \ldots, a_n; c)$ such that for all c-colorings of $[R]$ there exists a monochromatic solution to $a_1x_1 + \cdots + a_nx_n = 0$, where each $x_i \in [R]$.*

Proof. Define

$$s = \sum_{i \notin I} a_i$$

Choose $q \in I$ such that $|LCM(s, a_q)|$ is minimal, and let $u = LCM(s, a_q)$, where u is chosen to have the same sign as s.

We claim that if there exist positive integers a and d such that $a, \frac{ud}{s}, a - \frac{ud}{a_q} \in [R]$ are monochromatic, then there exists a monochromatic solution to the above equation. Namely,

$$x_i = \begin{cases} a - \frac{ud}{a_q} & \text{if } i = q \\ a & \text{if } i \neq q \in I \\ \frac{ud}{s} & \text{if } i \notin I \end{cases}$$

We can verify this as follows:

$$\begin{aligned}
\sum_{i=1}^{n} a_i x_i &= \sum_{i\in I} a_i x_i + \sum_{i\notin I} a_i x_i \\
&= \sum_{i\in I} a_i a - a_q \frac{ud}{b} + \sum_{i\notin I} a_i \frac{ud}{s} \\
&= 0 - a_q \frac{ud}{a_q} + s\frac{ud}{s} \\
&= 0
\end{aligned}$$

We apply Lemma 1 with $l = \frac{u}{s}$ and $m = -\frac{u}{a_q}$ to obtain an R large enough to guarantee the existence of a monochromatic triple $a, ld, a + md \in [R]$. Since u was chosen to have the same sign as s, l is guaranteed to be positive in our application of Lemma 1. If a_q also has the same sign as s then $m < 0$, whereas if a_q and s have opposite signs then $m > 0$. Formally, we have shown the following:

$$\begin{aligned}
R(a_1, \ldots, a_n; c) &\leq P\left(\frac{u}{s}, -\frac{u}{a_q}, c\right) \\
&= P\left(\frac{LCM\,(s, a_q)}{s}, -\frac{LCM\,(s, a_q)}{a_q}, c\right) \\
&= P\left(\frac{LCM\,\left(\sum_{i\notin I} a_i, a_q\right)}{\sum_{i\notin I} a_i}, -\frac{LCM\,\left(\sum_{i\notin I} a_i, a_q\right)}{a_q}, c\right)
\end{aligned}$$

$\square$

Definition 1. The Rado number $R(a_1, \ldots, a_n; c)$ is the smallest R such that for all c-colorings of $[R]$ there exists a monochromatic solution to $a_1x_1 + \cdots + a_nx_n = 0$.

The VDW proof of Lemma 1 gives the following upper bound on Rado numbers:

$$\begin{aligned}
R(a_1, \ldots, a_n; c) &\leq P\left(\frac{LCM\,\left(\sum_{i\notin I} a_i, a_q\right)}{\sum_{i\notin I} a_i}, -\frac{LCM\,\left(\sum_{i\notin I} a_i, a_q\right)}{a_q}, c\right) \\
&= P\left(\frac{u}{s}, -\frac{u}{a_q}, c\right) \\
&\leq f^{(c-1)}_{2,\frac{u}{s},-\frac{u}{a_q},c}\left(\max\left\{-\frac{u}{a_q} + 1, \frac{u}{s}\right\}\right)
\end{aligned}$$

The following sections will provide proofs of Lemma 1 for particular values of c, l, and m, resulting in better upper bounds on the Rado numbers associated with particular classes of equations.

4 The Easy Case: $l = 1$, $c = 2$

This section deals with the class of equations where, after forming $I \subseteq [n]$ such that $\sum_{i\in I} a_i = 0$, there exists a $q \in I$ such that a_q divides $s = \sum_{i\notin I} a_i$. For equations that fall into this category, $u = LCM(s, a_q) = s$. Therefore in the application of Lemma 1, $l = \frac{u}{s} = 1$ and $m = \frac{u}{a_q}$. In addition we restrict our attention to the two-color case.

Lemma 2 ($c = 2, l = 1$)**.** *For all $m \in \mathbb{N}$ and for all 2-colorings of $[1 + 3m + m^2]$ there exists a monochromatic triple $a, d, a + md \in [R]$.*

Proof. Our general approach is to do a case analysis of the potential colors that small numbers can take. There are two rules we use in this analysis. The first rule comes from taking $d = a$ in the above lemma.

Rule 1. *For any $a \in \mathbb{N}$, $a + am$ cannot be the same color as a, otherwise we are done. This is because $a, a, a + am$ would be a valid triple.*

The second rule is the more general case.

Rule 2. *For any $a, d \in \mathbb{N}$ that share a color, neither $a + md$ nor $d + ma$ can be that same color, otherwise we are done.*

Without loss of generality assume 1 is colored RED. By Rule 1 that means $1 + m$ must be BLUE, which means $(1 + m)^2 = 1 + 2m + m^2$ must be RED. Applying Rule 2 we get that $(1 + 2m + m^2) + m(1) = 1 + 3m + m^2$ must be BLUE.

CASE 1: 2 is RED. By Rule 2 that means $2 + m$ must be BLUE, which by Rule 2 means $(1 + m) + m(2 + m) = 1 + 3m + m^2$ must be RED. Since $1 + 3m + m^2$ must be either RED or BLUE, we are done.

CASE 2A: 2 is BLUE, 3 is RED. Since 2 is BLUE we can apply Rule 2 to it and $1 + m$ to conclude that $(1 + m) + m(2) = 1 + 3m$ is RED. However since 1 and 3 are RED, Rule 2 implies that $1 + 3m$ is BLUE, so we are done.

CASE 2B: 2 is BLUE, 3 is BLUE. Rule 2 implies $(1+m)+m(2) = 1+3m$ and $(1 + m) + m(3) = 1 + 4m$ must be RED, but applying Rule 2 to 1 and $1 + 3m$ implies $1 + 4m$ must be BLUE, so we are done with this case.

The result follows from the fact that $1+3m+m^2$ is greater than or equal to both $1+3m$ and $1+4m$ for $m \geq 1$.

$\square$

Example 1. $x+y-z=0$, $c=2$

Proof. In this case we choose $I = \{1,3\}$ since $a_1 + a_3 = 1 - 1 = 0$, and therefore $s = \sum_{i \notin I} a_i = a_2 = 1$. We choose $q = 3$ so $u = LCM(s, a_q) = LCM(1, -1) = 1$ (recall we always choose the sign of u to match that of s. Thus in our application of Lemma 1, $l = \frac{u}{s} = 1$ and $m = -\frac{u}{a_q} = 1$. Since $l = 1$, we can apply Lemma 2 to obtain an upper bound on the Rado number $R(1, 1, -1; 2) \leq 1 + 3m + m^2 = 5$. $\square$

At first it may seem that limiting ourselves to the class of equations where $l = 1$ is quite restrictive but in fact this class is quite large. We make the following informal comments about the size of this class:

1. There is no inherent bound on the number of variables or the size of the coefficients, since all that is required for $l = 1$ is that there exists a $q \in I$ such that a_q divides $s = \sum_{i \notin I} a_i$.

2. If there exists a $q \in I$ such that $a_q = \pm 1$, then the equation is in the class $l = 1$ regardless of any other coefficients. Thus we automatically get the following equations in this class: $x_1 + 2011x_2 - x_3 = 0$, $7x_1 - 6x_2 + 24x_3 - x_4 = 0$, and $5x_1 - x_2 - 4x_3 + \sum_{i=4}^{n} a_i x_i = 0$ (for arbitrary n and arbitrary a_i's).

3. Let a_q be the smallest coefficient (by absolute value) such that $q \in I$. If we choose a "random" equation, where we model $s = \sum_{i \notin I} a_i$ as being chosen uniformly at random from the set of integers (whatever that means), the probability that $l = 1$ is at least $\frac{1}{a_q}$ (since one in every a_q numbers is divisible by a_q). This is independent of the number of variables or the sizes of any other coefficients. In fact this probability will grow larger as the number of variables in I increases (and thus s has more numbers that can divide it).

5 The Case: $l = 1$, $c \geq 3$

The previous section gives a template for how to go about finding better upper bounds on Rado numbers. This can be generalized to $c \geq 3$ by changing the logic from "x cannot be RED, therefore it is BLUE" to "x cannot be RED, go on a keep coloring until $c-1$ colors have been ruled out for x." A particular branch of the search ends when c colors have been ruled out for a particular number. At this point the algorithm goes back and changes whatever its last decision was. This continues until all branches have been exhausted, at which point whichever branch gave the highest uncolorable number gives the upper bound.

This algorithm was coded up in C for general c and m (l is fixed to 1). The results, along with sample equations for which Rado's Theorem could be applied with the new bound, are given in the following table:

(c,m)	VDW-bound	new-bound	sample equation
(2,1)	W(3,2)	5	$x-y+z$
(2,2)	W(7,2)	11	$x-y+2z$
(2,3)	W(13,2)	19	$x-y+3z$
(2,4)	W(21,2)	49	$x-y+4z$
(2,5)	W(31,2)	101	$x-y+5z$
(3,1)	W(W(3,3)+1,3)	14	$x-y+z$
(3,2)	W(2W(7,3)+1,3)	75	$x-y+2z$
(3,3)	W(3W(13,3)+1,3)	253	$x-y+3z$
(4,1)	W(W(W(3,4)+1,4)+1,4)	61	$x-y+z$

References

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[3] R. Graham, B. Rothchild, and J. Spencer. *Ramsey Theory*. Wiley, 1990.

[4] B. Landmann and A. Robertson. *Ramsey Theory over the integers*. AMS, 2003.

[5] S. Shelah. Primitive recursive bounds for van der Waerden numbers. *Journal of the American Mathematical Society*, pages 683–697, 1988. `http://www.jstor.org/view/08940347/di963031/96p0024f/0`.

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