A Color Focusing Proof of the Hales-Jewett Theorem
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0.1 Introduction

Recall van der Waerden's Theorem

Theorem 0.1.1 *For all k, for all c, there exists $W = W(k,c)$ such that, for all* $\mathrm{COL}\colon [W] \to [c]$*, there exists $a, d \in \mathbb{N}$ such that*

$$a, a+d, \ldots, a+(k-1)d$$

are the same color.

In some textbooks and expositions of the proof of Theorem 0.1.1 the $W(3,2)$, $W(3,3)$, and $W(4,2)$ cases are presented and the reader is told (correctly) that they can do any particular case.

One way to prove Theorem 0.1.1 rigorously is to prove the Hales-Jewett Theorem, and then obtain Theorem 0.1.1 as an easy corollary. We will put off stating the Hales-Jewett Theorem until Section 0.4.

Bergelson and Leibman [1] proved the polynomial van der Waerden's theorem:

Theorem 0.1.2 *For all $p_1, \ldots, p_k \in \mathbb{Z}[x]$ such that $(\forall i)[p_i(0) = 0]$, for all c, there exists $W = W(p_1, \ldots, p_k; c)$ such that, for all* $\mathrm{COL}\colon [W] \to [c]$*, there exists $a, d \in \mathbb{N}$ such that*

$$a, a+p_1(d), \ldots, a+p_k(d)$$

are the same color.

The proof by Bergelson and Leibman [1] uses ergodic theory. Walters [11] provided a purely combinatorial proof of the polynomial van der Waerden theorem. He called his technique *color focusing*. In retrospect, the original proof of VDW's theorem could be phrased that way.

The original proof of the Hales-Jewett Theorem can also be phrased as a color-focusing argument.

We give a particularly explicit exposition of the standard color-focusing proof of the Hales-Jewett Theorem.

0.2 Examples of Points, Lines, Lines$^-$, and Completions

Let's look at all elements of $[3]^5$. There are 3^5 of them. Here they are:

$$\begin{array}{c} 11111 \\ 11112 \\ 11113 \\ 11121 \\ \vdots \\ 33333 \end{array}$$

We refer to these elements as *points*.

We give an example of a line and a line$^-$.

Here is a *line:*

$$\begin{aligned} \sigma_1 &= 11231 \\ \sigma_2 &= 12232 \\ \sigma_3 &= 13233 \end{aligned}$$

The first coordinate is constant $1-1-1$.

The second coordinate is increasing: $1-2-3$.

The third coordinate is constant: $2-2-2$.

The fourth coordinate is constant: $3-3-3$.

The fifth coordinate is increasing: $1-2-3$.

Let's look at the second coordinate. Note that

The second coordinate of σ_1 is 1.

The second coordinate of σ_2 is 2.

The second coordinate of σ_3 is 3.

Notation 0.2.1 We will usually write lines by listing the points vertically.

A *line*$^-$ is a line without the last point. The following is a line$^-$:

$$\begin{array}{c} 11231 \\ 12232 \end{array}$$

The *completion* of a line$^-$ is what would have been the last point.

If the alphabet has only 2 symbols then the completion might not be unique. Note that

11111 is a line$^-$.

The following are completions:

11112
11122
11222
12222
22222

If the alphabet has ≥ 3 symbols then the completion is unique. (We leave that as an easy exercise.)

Several lines$^-$ can have the same completion. Consider the following five lines$^-$ with $t = 3$.

line$^-$ 1	line$^-$ 2	line$^-$ 3	line$^-$ 4	line$^-$ 5
11111	31111	33111	33311	33331
22222	32222	33222	33322	33332

Note that 33333 is a completion for all five of these line$^-$.

0.3 Formal Definitions of Points, Lines, Line$^-$, and Completions

Notation 0.3.1 Throughout this paper $\mathbb{N} = \{1, 2, \ldots\}$ so it does not include 0.

Def 0.3.2 Let $t \in \mathbb{N}$. t will be the size of our alphabet. Let $N \in \mathbb{N}$. We will be looking at $[t]^N$.

1. A *point* is an element of $[t]^N$.
2. A *line* is a sequence of t points $\sigma_1, \sigma_2, \ldots, \sigma_t$ together with a set $I \subseteq [N]$ such that the following holds.
   (a) $I \neq \emptyset$.
   (b) For all $i \in I$ and all $j \in [t]$, the ith symbol of σ_j is j.
   (c) For all $i \notin I$ there exists $s \in [t]$ such that, for all $j \in [t]$, the ith symbol of σ_j is s.
3. A *line*$^-$ is a line without the last point.
4. A *Completion of a line*$^-$ L^- is the would-be t^{th} point of L^-. If $t = 2$ then there may be many completions. If $t \geq 3$ then the completion is unique.
5. Let COL: $[t]^N \to [c]$. A *monochromatic line* is a line where all t points are the same color. Henceforth we will use the phrase *mono line*. We may specify the coloring by saying *mono line via* COL.
6. Let COL: $[t]^N \to [c]$. A *monochromatic line*$^-$ is a line$^-$ where all $t-1$ points are the same color. Henceforth we will use the phrase *mono line*$^-$. We may specify the coloring by saying *mono line*$^-$ *via* COL.

0.4 Statement of the Hales-Jewett Theorem and Some Easy Cases

The following is the Hales-Jewett Theorem. It was first proven by Hales and Jewett [4].

Theorem 0.4.1 *Let $t, c \in \mathbb{N}$. There exists $HJ = HJ(t, c)$ such that, for all* $\mathrm{COL}\colon [t]^{\mathrm{HJ}(t,c)} \to [c]$ *there exists a mono line.*

Notation 0.4.2 We will use the notation $\mathrm{HJ}(t, c)$ for two items:

- $\mathrm{HJ}(t, c)$ is the least number such that the theorem holds. For example, we might write (this is not known)

$$\mathrm{HJ}(3, 8) \le 10^{10}.$$

- $\mathrm{HJ}(t, c)$ means the Hales-Jewett Theorem is true for parameters t, c. For example, we might write (this is not known)

$$\mathrm{HJ}(t-1, 2^c) \to \mathrm{HJ}(t, c).$$

  We will use $\mathrm{HJ}(t, \omega)$ to mean $(\forall c)[\mathrm{HJ}(t, c)]$. For example, we might write (this is true and is most of the proof)

$$(\forall t, c \in \mathbb{N})[\mathrm{HJ}(t-1, \omega) \to \mathrm{HJ}(t, c)].$$

The meaning of $\mathrm{HJ}(t, c)$ will be clear from context.

Easy Cases of the Hales-Jewett Theorem

Lemma 0.4.3 *Let $t \in \mathbb{N}$.* $\mathrm{HJ}(t, 1)$ *and* $\mathrm{HJ}(2, \omega)$ *are true.*

Proof:
1) $\mathrm{HJ}(t, 1) = 1$. Any 1-coloring of $[t]^1 = [t]$ easily has a mono line. For example, if we 1-color [4] we have that (1), (2), (3), (4) are all RED and they form a line.

2) For all $c \in \mathbb{N}$, $\mathrm{HJ}(2, c) \le c$.

Let $\mathrm{COL}\colon [2]^c \rightarrow [c]$.

Consider the $c+1$ points

$1\cdots 111$ (c 1's, zero 2's)

$1\cdots 112$ ($c-1$ 1's, one 2)

$1\cdots 122$ ($c-2$ 1's, two 2's)

$\cdots$

$2\cdots 222$ (zero 1's, c 2's).

Since these points are c colored, there must be two points that are the same color. These two points form a mono line. We just showed $(\forall c)[\mathrm{HJ}(2,c)]$. Hence we just showed $\mathrm{HJ}(2,\omega)$.

We leave it as an exercise to show that $\mathrm{HJ}(2,c)=c$; however, that is not needed for the proof. ▮

We list an obvious statement to make sure we are all on board with the notation.

Obvious Fact 1 Assume the number $\mathrm{HJ}(t,c)$ exists. Let

$$\mathrm{COL}\colon [t]^{\mathrm{HJ}(t,c)} \rightarrow [c].$$

There will be a mono line.

Why? Because $\mathrm{HJ}(t,c)$ is the number x such that, for all

$$\mathrm{COL}\colon [t]^x \rightarrow [c]$$

there is a mono line. So a mono line exists by the very definition of $\mathrm{HJ}(t,c)$.

0.5 The Key Lemma

We prove a lemma from which the Hales-Jewett Theorem will follow. We defer the (easy) proof that the lemma implies the Hales-Jewett Theorem until after the proof of the lemma.

Lemma 0.5.1 *Fix $t, c \in \mathbb{N}$ with $t \geq 3$. Assume* $\mathrm{HJ}(t-1, \omega)$. *Then, for all r, there exists $U(r)$ such that, for all* $\mathrm{COL}\colon [t]^{U(r)} \to [c]$ *one of the following statements holds. (Formally we should write $U(t, c, r)$. We will use this notation when we prove that the Key Lemma implies the Hales-Jewett Theorem.)*

Statement I: *There exists a mono line $L \subseteq [t]^{U(r)}$.*

Statement II: *There exist r mono lines$^-$ $L_1^-, L_2^-, \ldots, L_r^- \subseteq [t]^{U(r)}$, and a point $Q \in [t]^{U(r)}$, such that the following hold:*

- *The r lines$^-$ $L_1^{,} \ldots, L_r^-$ have pairwise distinct colors.*
- *Q is the completion of all of the L_i^-'s.*
- *Q has a color that is different from the r colors of $L_1^-, L_2^-, \ldots, L_r^-$.*

When such a number exists (which we will prove), let $U(r)$ be the least one.

Proof:

We make an obvious statement similar to Obvious Fact 1 to make sure we are all on the same page.

Obvious Fact 2 Assume the number $U(r)$ exists. Let

$$\mathrm{COL}\colon [t]^{U(r)} \to [c].$$

Then there will either be a mono line (Statement I) or r mono lines$^-$ that (1) have different colors, (2) have the same completion, and (3) the completion has a color different from the r lines$^-$ (Statement II).

Why? because $U(r)$ is the least number x such that, for all

$$\mathrm{COL}\colon [t]^x \to [c]$$

either Statement I or Statement II holds.

The proof of the lemma is by induction.

Base Case $r = 1$. We claim $U(1) \le \mathrm{HJ}(t-1, c)$. Note that we know $\mathrm{HJ}(t-1, c)$ exists.

Let $\mathrm{COL}\colon [t]^{\mathrm{HJ}(t-1,c)} \to [c]$.

Let $\mathrm{COL}'\colon [t-1]^{\mathrm{HJ}(t-1,c)} \to [c]$ be the restriction of COL to $[t-1]^{\mathrm{HJ}(t-1,c)}$. By Obvious Fact 1 there is a mono line L via COL'; however, L is of length $t-1$ and uses the symbols $\{1, \ldots, t-1\}$.

L is a mono line in $[t-1]^{\mathrm{HJ}(t-1,c)}$ and hence L is a mono line$^-$ in $[t]^{\mathrm{HJ}(t-1,c)}$. Assume the color is RED.

If the completion is colored RED then we have a mono line via COL, so Statement I is true.

If the completion is not colored RED then we have $r = 1$ mono line$^-$ via COL with a completion of a different color, so Statement II is true.

Note 0.5.2 Up to this point everything we did was easy. Now it gets hard.

Induction Hypothesis $U(r)$ exists.

Induction Step

Let $X = c^{t^{U(r)}}$.

Note that X is the number of ways to c-color $[t]^{U(r)}$. Indeed, we will use X to denote both the number of such colorings and the set of all such colorings. X stands for eXtremely large.

We will show that

$$U(r+1) \le \mathrm{HJ}(t-1, X) + U(r).$$

Let $\mathrm{COL}\colon [t]^{\mathrm{HJ}(t-1,X)+U(r)} \to [c]$.

We view

$$[t]^{\mathrm{HJ}(t-1,X)+U(r)}$$

as

$$[t]^{\mathrm{HJ}(t-1,X)} \times [t]^{U(r)}.$$

We define a coloring COL':

$$\mathrm{COL}'\colon [t-1]^{HJ(t-1,X)} \to X.$$

Note 0.5.3

1. We will be interpreting X as the set of c-coloring of $[t]^{U(r)}$. So we will be coloring elements of $[t-1]^{\mathrm{HJ}(t-1,X)}$ with a coloring.

2. Even before defining COL' we know that the coloring will have a mono line (of length $t-1$) by Obvious Fact 1.

Given $\sigma \in [t-1]^{HJ(t-1,X)}$, $\mathrm{COL}'(\sigma)$ is the following c-coloring of $[t]^{U(r)}$:

$$\chi_\sigma(\tau) = \mathrm{COL}(\sigma\tau).$$

By Obvious Fact 1 there exists a mono line via COL'. Hence there exists $t-1$ elements of $[t]^{\mathrm{HJ}(t-1,X)}$ that all map to the same coloring. Let the elements be:

σ_1
σ_2
$\vdots$
σ_{t-1}

Let χ be the common COL'-color of $\sigma_1, \ldots, \sigma_{t-1}$.

Since $\sigma_1, \ldots, \sigma_{t-1}$ is a line in $[t-1]^{\mathrm{HJ}(t-1,X)}$ it is a line$^-$ in $[t]^{\mathrm{HJ}(t-1,X)}$. Let σ be the completion of that line$^-$.

Back to χ. Recall that $\chi\colon [t]^{U(r)} \to [c]$. By Obvious Fact 2 one of the following occurs.

Statement I There is a mono line via χ. Let that line be

τ_1
τ_2
$\vdots$
τ_t.

Assume that all of these points have color (from χ) R. Then

$$(\forall 1 \le i \le t)[R = \chi(\tau_i) = \mathrm{COL}(\sigma_1\tau_i)].$$

Hence the following line is mono via COL:
$\sigma_1\tau_1$
$\sigma_1\tau_2$
$\vdots$
$\sigma_1\tau_t$.

Since we have a mono line via COL, our Statement I is true.

Statement II There are r mono line^{-} via χ which are all different colors, and all have the same completion, which is itself a different color.

Let the r mono line^{-} via χ be

$$\begin{array}{cccc}
\tau_1^1 & \tau_1^2 & \cdots & \tau_1^r \\
\tau_2^1 & \tau_2^2 & \cdots & \tau_2^r \\
\tau_3^1 & \tau_3^2 & \cdots & \tau_3^r \\
\vdots & \vdots & \vdots & \vdots \\
\tau_{t-1}^1 & \tau_{t-1}^2 & \cdots & \tau_{t-1}^r \\
\hline
c_1 & c_2 & \cdots & c_r
\end{array}$$

Let τ be the completion of all of them. Let $\chi(\tau) = c_{r+1}$. As part of Statement II we know that $c_{r+1} \notin \{c_1, \ldots, c_r\}$.

Note that, for all $1 \le i, j \le t-1$, and all $1 \le k \le r$,

$$\mathrm{COL}(\sigma_i \tau_j^k) = \chi(\tau_j^k) = c_k.$$

In particular, for $1 \le i \le r$ and $1 \le j \le t-1$, $\mathrm{COL}(\sigma_j \tau_i^j) = c_i$.

The following r lines^{-} via COL have colors $c_1, \ldots, c_r$ and common completion $\sigma\tau$.

$$\begin{array}{cccc}
\sigma_1\tau_1^1 & \sigma_1\tau_1^2 & \cdots & \sigma_1\tau_1^r \\
\sigma_2\tau_2^1 & \sigma_2\tau_2^2 & \cdots & \sigma_2\tau_2^r \\
\sigma_3\tau_3^1 & \sigma_3\tau_3^2 & \cdots & \sigma_3\tau_3^r \\
\vdots & \vdots & \vdots & \vdots \\
\sigma_{t-1}\tau_{t-1}^1 & \sigma_{t-1}\tau_{t-1}^2 & \cdots & \sigma_{t-1}\tau_{t-1}^r \\
\hline
c_1 & c_2 & \cdots & c_r
\end{array}$$

Let $\mathrm{COL}(\sigma\tau) = c$.

Case 1 There exists $1 \le i \le r$ such that $c = c_i$.

Then we have a mono line:

$$\begin{array}{c}
\sigma_1\tau_1^i \\
\sigma_2\tau_2^i \\
\sigma_3\tau_3^i \\
\vdots \\
\sigma_{t-1}\tau_{t-1}^i \\
\sigma\tau
\end{array}$$

Case 2 $c \notin \{c_1, \ldots, c_r\}$. We obtain one more mono line$^-$ by using the common completion τ in the suffix. Consider L^-:

$$\begin{array}{c} \sigma_1\tau \\ \sigma_2\tau \\ \sigma_3\tau \\ \vdots \\ \sigma_{t-1}\tau \end{array}$$

Since, for all $1 \le i \le t-1$, $\mathrm{COL}(\sigma_i\tau) = \chi(\tau) = c_{r+1}$, L^- is mono of color c_{r+1}. Its completion is $\sigma\tau$, the same completion as the previous r lines.
Case 2.1: If $c = c_{r+1}$ then L^- together with $\sigma\tau$ is a mono line, so Statement I holds.

Case 2.2: If $c \ne c_{r+1}$ then the $r+1$ mono lines$^-$ (1) have distinct colors $c_1, \ldots, c_{r+1}$, (2) have common completion $\sigma\tau$, and (3) the completion has a color different from all $r+1$ colors. Hence Statement II holds. ∎

0.6 The Key Lemma Implies HJ

We restate, and then prove, the Hales-Jewett Theorem.

Theorem 0.6.1 *Let $t, c \in \mathbb{N}$. There exists $HJ = HJ(t,c)$ such that, for all* $\mathrm{COL}\colon [t]^{\mathrm{HJ}(t,c)} \to [c]$ *there exists a mono line.*

Proof: We prove this by induction on t.
Base Case: $t = 1$. By Lemma 0.4.3, $\mathrm{HJ}(1,\omega)$ is true.
Base Case: $t = 2$. By Lemma 0.4.3, $\mathrm{HJ}(2,\omega)$ is true.
Induction Hypothesis $\mathrm{HJ}(t-1,\omega)$.
Induction Step By the IH we have the premise for the Key Lemma. We take the $r = c$ case. So $U(t,c,c)$ exists. We show $\mathrm{HJ}(t,c) \le U(t,c,c)$.
Let $\mathrm{COL}\colon [t]^{U(t,c,c)} \to [c]$. By the Key Lemma either
Statement I: There exists a mono line $L \subseteq [t]^{U(t,c,c)}$ via COL.
Statement II: There exist c mono lines$^-$ $L_1^-, L_2^-, \ldots, L_c^- \subseteq [t]^{U(t,c,c)}$, each of a different color, and a point $Q \in [t]^{U(t,c,c)}$, such that (1) Q is the completion of every L_i^- and (2) the color of Q differs from the color of every L_i^-.

If Statement II holds then there are c mono lines$^-$ with different colors, and also a point with a color that is not one of those c. Hence the c lines$^-$

use all c colors, while Q has a color different from all of them, requiring a $(c+1)$st color. This is impossible. Therefore Statement II cannot be true, so Statement I is true. Hence there is a mono line via COL. ∎

0.7 How Fast Does $\mathrm{HJ}(t,c)$ Grow?

We use Knuth's arrow notation in this section.

From the proof we gave of the Hales-Jewett Theorem, which is essentially what Hales and Jewett did in 1963, one can obtain:

$$\mathrm{HJ}(t,c) \le 2 \uparrow^{t-1} (2 \uparrow^{t-2} c).$$

Note that since the number of arrows increases with t, the bound is not primitive recursive. The upper bound on $\mathrm{HJ}(t,c)$ is roughly an Ackermann-type growth. n It was an open question as to whether $\mathrm{HJ}(t,c)$ could be bounded by a primitive recursive function.

In 1988 Shelah [9] (see also [7]) showed that $\mathrm{HJ}(t,c)$ has a primitive recursive upper bound. Informally, his bound involves enormous *supertowers*; not merely towers of exponentials, but iterations of the operation forming such towers.

In 2026 Shelah [10] substantially improved the upper bound. He reduced the upper bound from these iterated supertower constructions to an *ordinary tower of exponentials.* The size of the tower bounding $\mathrm{HJ}(t,c)$ depends on t. Essentially, the dependence on c is in the base of the tower.

0.8 HJ Implies van der Waerden's Theorem

Theorem 0.8.1 *(van der Waerden's Theorem) For all k, for all c, there exists $W = W(k,c)$ such that, for all $\mathrm{COL}\colon [W] \to [c]$, there exists $a, d \in \mathbb{N}$ such that*

$$a, a+d, \ldots, a+(k-1)d$$

are the same color.

Proof:

We show that $W(k,c) \le k\mathrm{HJ}(k,c)$.

Let $H = \mathrm{HJ}(k,c)$.

Let COL: $[kH] \rightarrow [c]$. Define COL$'$: $[k]^{\mathrm{HJ}(k,c)} \rightarrow [c]$ as

$$\mathrm{COL}'(a_1, \ldots, a_H) = \mathrm{COL}(\sum_{i=1}^{H} a_i).$$

Since $(\forall 1 \le i \le H)[a_i \le k]$, we are feeding into COL numbers that are $\le k\mathrm{HJ}(k,c)$.

By the Hales-Jewett Theorem there is a mono line via COL$'$. We give an example of how from this line we can obtain a mono k-AP.

Say the line is

11112334. The sum of the coordinates is 16.
21212334. The sum of the coordinates is 18.
31312334. The sum of the coordinates is 20.
41412334. The sum of the coordinates is 22.

Then (16,18,20,22) is a mono 4-AP.

We leave it to the reader to prove formally that a mono line via COL$'$ leads to a mono k-AP via COL. ∎

0.9 HJ implies the Square Theorem

Theorem 0.9.1 *For all c there exists $N = N(c)$ such that, for all* COL$[N] \times [N] \rightarrow [c]$ *there exist four points that are the same color that form an axis-parallel square. Formally, there exists $a, b, d \in \mathbb{N}$ such that*

$$\mathrm{COL}(a,b) = \mathrm{COL}(a+d,b) = \mathrm{COL}(a,b+d) = \mathrm{COL}(a+d,b+d).$$

Proof:

For this application we use an alphabet that is not a set of numbers. Let $\Sigma = \{(0,0),(0,1),(1,0),(1,1)\}$. The ordering on Σ is

$$(0,0) < (0,1) < (1,0) < (1,1).$$

Since $|\Sigma| = 4$ we will be using $t = 4$.

We show $N(c) \le \mathrm{HJ}(4,c) + 1$.

Let COL: $[\mathrm{HJ}(4,c)+1] \times [\mathrm{HJ}(4,c)+1] \rightarrow [c]$.

We define a coloring $\mathrm{COL}'\colon \{(0,0),(0,1),(1,0),(1,1)\}^{\mathrm{HJ}(4,c)} \to [c]$.

$$\mathrm{COL}'((a_1,b_1),\ldots,(a_H,b_H)) = \mathrm{COL}((1+\sum_{i=1}^{H} a_k),(1+\sum_{i=1}^{H} b_i)).$$

(We need the 1 since the domain of COL does not include ordered pairs where either coordinate is 0.)

Since $(\forall 1 \le i \le H)[a_i, b_i \le 1]$ we are feeding into COL pairs in $[\mathrm{HJ}(4,c)+1] \times [\mathrm{HJ}(4,c)+1]$.

By the Hales-Jewett Theorem there is a mono line via COL'. Note that it is of length four.

We give an example of how from this line we can obtain a mono square.

Say the line is

$$(0,0),(0,0),(0,0),(0,0)(1,0),(1,0),(1,0),(0,1),(0,1)(1,1),(1,1).$$
The sum plus (1,1) gives $(6,5)$.
$$(0,1),(0,0),(0,1),(0,1)(1,0),(1,0),(1,0),(0,1),(0,1)(1,1),(1,1).$$
The sum plus (1,1) gives $(6,8)$.
$$(1,0),(0,0),(1,0),(1,0)(1,0),(1,0),(1,0),(0,1),(0,1)(1,1),(1,1).$$
The sum plus (1,1) gives $(9,5)$.
$$(1,1),(0,0),(1,1),(1,1)(1,0),(1,0),(1,0),(0,1),(0,1)(1,1),(1,1).$$
The sum plus (1,1) gives $(9,8)$.

Then (6,5), (6,8), (9,5), (9,8) is a mono square.

We leave it to the reader to prove formally that a mono line via COL' leads to a mono square via COL. ∎

0.10 HJ implies the Two-Dimensional Gallai-Witt Theorem

We just do the $k = 3$ version of the Two-Dimensional Gallai-Witt Theorem.

Theorem 0.10.1 *For all c there exists $N = N(c)$ such that, for all $\mathrm{COL}[N]\times[N] \to [c]$ there exists a mono regular 3×3 grid. Formally, there exists $a, b, d \in \mathbb{N}$ such that*

$$\mathrm{COL}(a,b) = \mathrm{COL}(a+d,b) = \mathrm{COL}(a+2d,b)$$

$$= \mathrm{COL}(a,b+d) = \mathrm{COL}(a+d,b+d) = \mathrm{COL}(a+2d,b+d)$$

$$= \mathrm{COL}(a,b+2d) = \mathrm{COL}(a+d,b+2d) = \mathrm{COL}(a+2d,b+2d).$$

Proof:

For this application we use an alphabet that is not a set of numbers. Let $\Sigma = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}$.
The ordering on Σ is

$$(0,0) < (0,1) < (0,2) < (1,0) < (1,1) < (1,2) < (2,0) < (2,1) < (2,2).$$

Since $|\Sigma| = 9$ we will be using $t = 9$.
We show $N(c) \le 2\mathrm{HJ}(9,c)+1$. Let $H = \mathrm{HJ}(9,c)$.
Let $\mathrm{COL}\colon [2H+1] \times [2H+1] \to [c]$.
We define a coloring
$\mathrm{COL}'\colon \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}^H \to [c]$.

$$\mathrm{COL}'((a_1,b_1),\ldots,(a_H,b_H)) = \mathrm{COL}((1+\sum_{i=1}^{H} a_k, 1+\sum_{i=1}^{H} b_i)).$$

(We need the 1 since the domain of COL does not include ordered pairs where either coordinate is 0.)

Since $(\forall 1 \le i \le H)[a_i, b_i \le 2]$ we are feeding into COL pairs in $[2H+1] \times [2H+1]$.

By the Hales-Jewett Theorem there is a mono line via COL$'$. Note that it is of length nine.

We leave it to the reader to show that, from this mono line via COL$'$, we can obtain a mono 3×3 grid via COL.

∎

0.11 HJ implies the Gallai-Witt Theorem

We state the Gallai-Witt Theorem and leave it to the reader to prove it.

Theorem 0.11.1 *Let $d, k, c \in \mathbb{N}$. There exists $G = G(d, k, c)$ such that, for all* COL: $[G]^d \to [c]$ *there exists a mono regular $k \times \cdots k$ (d times) grid.*

0.12 Known HJ numbers

We give a table of known HJ numbers, and known upper and lower bounds on HJ numbers. We have some redundant results to show the historical progression. Spoiler alert: not much is known.

Result	Reference
$\mathrm{HJ}(2, c) = c$	Easy and in this paper
$\mathrm{HJ}(3, 2) = 4$	Hindman and Tressler [5]
$\mathrm{HJ}(4, 2) \leq 10^{11}$	Lavrov [6]
$\mathrm{HJ}(4, 2) \geq 12$	N. Conlon and Farnsworth and Robertson [3]
$\mathrm{HJ}(4, 2) \geq 14$	Mouhib and Halbeisen [8]
$\mathrm{HJ}(3, 3) \leq 3^{36} + 732 \sim 10^{17.2}$	D. Conlon [2]
$\mathrm{HJ}(3, 3) \geq 13$	N. Conlon and Farnsworth and Robertson [3]
$\mathrm{HJ}(3, 3) \geq 22$	Mouhib and Halbeisen [8]

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