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A RESTRICTED VERSION OF HALES–JEWETT'S THEOREM

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1.

A well known theorem of van der Waerden states that for every pair δ, k of positive integers there exists a positive integer n with the property that for every partition of $\{1, \dots, n\}$ into δ many classes there exists a k-term arithmetic progression contained in one class. Thus in order to obtain a k-term arithmetic progression within one class of the partition a much richer structure (viz an n-term progression) is partitioned. Erdős [1] conjectured that for every pair δ, k of positive integers there exists a set A of positive integers which contains no $(k + 1)$-term progression and still has the property that for every partition of A into δ many classes at least one of the classes contains a k-term progression. That such a restricted version of van der Waerdens theorem is valid was shown independently by Spencer [8] with the aid of Hales – Jewett's theorem [4] and Nešetřil – Rödl [6] by a direct construction. In this paper we shall give a restricted version of Hales – Jewett's theorem for partitioning 0 parameter sets.

2.

Let A be a finite alphabet. A^n is the set of words of length n over A. A^n may also be viewed as an n-dimensional cube over A. A k-dimensional subcube C of A^n has a parametric representation of the following type

$$C = \{(a_0, \ldots, \lambda_0, \ldots, \lambda_1, \ldots, \lambda_{k-1}, \ldots, a_{n-1}) \mid \lambda_i \in A,\ i < k\}.$$

This representation is given by the word

$$f = (a_0, \ldots, \lambda_0, \ldots, \lambda_1, \ldots, \lambda_{k-1}, \ldots, a_{n-1}).$$

This suggests the following definition:

Definition 2.1. Let A be a finite alphabet and n, k be nonnegative integers. The set $[A]\binom{n}{k}$ of *k-parameter words of length n over A* is the set of all mappings $f\colon n \to A \cup \{\lambda_i \mid i < k\}$ where $A \cap \{\lambda_i \mid i < k\} = \phi$ satisfying

2.1.1. For every $j < k$ there exists $i < n$ such that $f(i) = \lambda_j$.

2.1.2. $\min f^{-1}(\lambda_i) < \min f^{-1}(\lambda_j)$ whenever $i < j < k$.

2.1.1 guarantees that k-parameters really occur in f and 2.1.2 gives a natural ordering of the first occurrences of the parameters. With these conditions we obtain a bijection between the k-parameter words of length n over A and the k-dimensional subcubes of A^n. Next we define the composition of parameter words.

Definition 2.2. Let $f \in [A]\binom{n}{m}$ and $g \in [A]\binom{m}{k}$. The k-parameter word $f \cdot g \in [A]\binom{n}{k}$ is defined by

$$f \cdot g(i) = \begin{cases} f(i) & \text{if} \quad f(i) \in A \\ g(j) & \text{if} \quad f(i) = \lambda_j. \end{cases}$$

In geometric terms $f \cdot g$ is the k-dimensional subcube of the m-dimensional subcube f which has parametric representation g in A^m.

Theorem (Hales – Jewett [4]). *Let* A *be a finite alphabet. For every pair* m, δ *of positive integers there exists a positive integer* n *such that for every coloring* $\Delta\colon [A]\binom{n}{0} \to \delta$ *of the points of the* n*-dimensional cube over* A *there exists a monochromatic* $f \in [A]\binom{n}{m}$, *i.e. the coloring* $\Delta_f\colon [A]\binom{m}{0} \to \delta$ *defined by* $\Delta_f(g) = \Delta(f \cdot g)$ *is constant.*

3.

Here we prove a restricted version of Hales–Jewett's theorem: For positive integers δ, m there exists a positive integer n and a subset S of $[A]\binom{n}{0}$ such that S does not contain an $(m+1)$-dimensional subcube, i.e. for every $f \in [A]\binom{n}{m+1}$ there exists $g \in [A]\binom{m+1}{0}$ such that $f \cdot g \notin S$, and still for every coloring of the points of S there exists a monochromatic m-dimensional subcube in S.

Notation 3.1. Let $S \subseteq [A]\binom{n}{0}$ and $k < n$.

$$\mathscr{F}_k(S) := \left\{ f \in [A]\binom{n}{k} \mid f \cdot [A]\binom{k}{0} \subseteq S \right\} =$$

$$= \left\{ f \in [A]\binom{n}{k} \mid f \cdot g \in S \text{ for every } g \in [A]\binom{k}{0} \right\}.$$

$\mathscr{F}_k(S)$ is the set of all k-dimensional subcubes of A^n which are contained in S. In particular $\mathscr{F}_0(S) = S$.

We are ready to state the main theorem.

Main theorem. *Let* A *be a finite alphabet and* δ, m *be positive integers. Then there exists a positive integer* n *and* $S \subseteq [A]\binom{n}{0}$ *such that*

(α) $\mathscr{F}_{m+1}(S) = \phi$, *i.e.* S *does not contain* $(m+1)$*-dimensional subcubes.*

(β) *For every coloring* $\Delta\colon S \to \delta$ *of the points of* S *with* δ *many colors there exists* $f \in \mathscr{F}_m(S)$ *such that the coloring* $\Delta_f\colon [A]\binom{m}{0} \to \delta$ *defined by* $\Delta_f(g) = \Delta(f \cdot g)$ *is constant.*

Notation 3.2. A convenient abbreviation for (α) and (β) is "$S \xrightarrow{A} (m)_\delta$".

The crucial part of the proof of the main theorem consists in showing its validity for $m = 1$. This case is stated in

Lemma 3.3. *Let A be a finite alphabet and δ be a positive integer. There exists a positive integer n and a subset S of $[A]\binom{n}{0}$ such that $S \xrightarrow{A} (1)_\delta$.*

The main theorem follows from this lemma by concatenation of various sets S.

Definition 3.4. Let $f \in [A]\binom{m}{0}$ and $g \in [A]\binom{n}{0}$. The concatenation $f \otimes g \in [A]\binom{m+n}{0}$ is defined by

$$f \otimes g(i) = \begin{cases} f(i) & \text{for} \quad i < m \\ g(i-m) & \text{for} \quad m \leqslant i < m+n. \end{cases}$$

Thus $f \otimes g$ is the usual concatenation of words.

Next we derive the Main theorem from Lemma 3.3. Proceed by induction on m. $m = 1$ is settled by the lemma. For $m + 1$ consider $S_0 \subseteq [A]\binom{n_0}{0}$ with $S_0 \xrightarrow{A} (m)_\delta$, which exists by induction hypothesis. Lemma 3.3 guarantees the existence of a positive integer n_1 and $S_1 \subseteq$ $\subseteq [A]\binom{n_1}{0}$ with $S_1 \xrightarrow{A} (1)_{\delta'}$, where $\delta' = \delta^{|S_0|}$.

Claim. $S_0 \otimes S_1 \xrightarrow{A} (m+1)_\delta$ *where* $S_0 \otimes S_1 = \{g \otimes h \mid g \in S_0, h \in S_1\}$.

We have to check conditions (α) and (β) of the Main theorem:

For (α): Let $f \in [A]\binom{n_0 + n_1}{m+2}$.

In case $\min f^{-1}(\lambda_m) < n_0$, consider $f^* \in [A]\binom{n_0}{m+1}$ defined by

$$f^*(i) = \begin{cases} f(i) & \text{for} \quad i < n_0 \quad \text{and} \quad f(i) \neq \lambda_{m+1} \\ \lambda_0 & \text{for} \quad i < n_0 \quad \text{and} \quad f(i) = \lambda_{m+1}. \end{cases}$$

Since $f^* \notin \mathscr{F}_{m+1}(S_0) = \phi$ there exists $g \in [A]\binom{m+1}{0}$ such that $f^* \cdot g \notin S_0$. Therefore $f \notin \mathscr{F}_{m+2}(S_0 \otimes S_1)$.

In case $\min f^{-1}(\lambda_m) \geqslant n_0$ consider $f^* \in [A]\binom{n_1}{2}$ defined by

$$f^*(i) = \begin{cases} f(i+n_0) & \text{if} \quad f(i+n_0) \in A \\ \lambda_0 & \text{if} \quad f(i+n_0) \in \{\lambda_0, \ldots, \lambda_m\} \\ \lambda_1 & \text{if} \quad f(i+n_0) = \lambda_{m+1}. \end{cases}$$

Again since $f^* \notin \mathscr{F}_2(S_1) = \phi$, there exists $g \in [A]\binom{2}{0}$ such that $f^* \cdot g \notin S_1$ and therefore $f \notin \mathscr{F}_{m+2}(S_0 \otimes S_1)$.

For (β): Let $\Delta\colon S_0 \otimes S_1 \to \delta$ be a coloring. Consider first the coloring $\Delta_1\colon S_1 \to \delta^{|S_0|}$ defined by $\Delta_1(\xi) = \Delta(g \otimes \xi \mid g \in S_0)$. By choice of S_1 there exists $h \in \mathscr{F}_1(S_1)$ such that Δ_1 is constant on h. Consider next $\Delta_0\colon S_0 \to \delta$ defined by $\Delta_0(\eta) = \Delta(\eta \otimes ha)$, where $a \in [A]\binom{1}{0}$ may be chosen arbitrarily. By choice of S_0 there exists a $g \in \mathscr{F}_m(S_0)$ such that Δ_0 is constant on g. Thus Δ is constant on $g \mathbin{\tilde{\otimes}} h \in$ $\in \mathscr{F}_{m+1}(S_0 \otimes S_1)$ where

$$g \mathbin{\tilde{\otimes}} h(i) = \begin{cases} g(i) & \text{for} \quad i < n_0 \\ \lambda_m & \text{for} \quad i \geqslant n_0 \quad \text{and} \quad h(i-n_0) = \lambda_0 \\ h(i-n_0) & \text{for} \quad i \geqslant n_0 \quad \text{and} \quad h(i-n_0) \in A. \blacksquare \end{cases}$$

Proof of Lemma 3.3. Proceed by induction on $|A|$. For $|A| = 1$ the lemma is obvious. For $|A| = 2$, e.g. $A = \{0, 1\}$, choose $n = \delta$ and consider the following set $S \subseteq [\{0, 1\}]\binom{n}{0}$:

$$S := \left\{\begin{matrix} (0 \ldots\ldots\ldots 0), \\ (0 \ldots\ldots 0\ \ 1), \\ (0 \ldots 0\ \ 1\ \ 1), \\ \vdots \qquad\quad \vdots \\ (1 \ldots\ldots 1\ \ 1), \end{matrix}\right\}$$

Obviously S does not contain a 2-dimensional subcube because each element of S has the property that all entries after the first occurrence of the letter 1 are 1, i.e. no 0 occurs. Thus for every $f \in [\{0, 1\}]\binom{n}{2}$ we have that $f(1, 0) \notin S$. Furthermore, any two elements of S form a 1-dimensional subcube. As $|S| = n + 1 = \delta + 1$ the pigeon hole principle guarantees the existence of two elements of S having the same color for any coloring $\Delta\colon S \to \delta$.

The general step may be done by a somewhat tricky iteration of the ideas just mentioned. Consider the alphabet $A = \{0, \ldots, t-1, t\}$. By induction hypothesis applied to $A^* = \{o, \ldots, t-1\}$, for every δ^* there exists n and $S \subseteq [A]\binom{n}{0}$ satisfying $S \xrightarrow{A^*} (1)_{\delta^*}$. This guarantees the existence of positive integers $n_0, \ldots, n_{\delta-1}$ and sets $S_i \subseteq$ $\subseteq [A^*]\binom{n_i}{0}$ $(i < \delta)$ satisfying

$$S_i \xrightarrow{A^*} (1)_{\delta_i}$$

where $\delta_0 = \delta^\delta$ and $\delta_{i+1} = \delta^{(\delta-1-i)\cdot|S_i \times S_{i-1} \times \ldots \times S_0|}$ for $i < \delta - 1$.

Let $n = n_0 + \ldots + n_{\delta-1}$. In order to define $S \subseteq [A]\binom{n}{0}$ consider the following scheme:

$$T = \underbrace{\begin{matrix} A^* \times \ldots\ldots\ldots \times A^* \cup \\ A^* \times \ldots \times A^* \times \{t\} \cup \\ A^* \times \ldots \times \{t\} \times \{t\} \cup \\ \vdots \qquad\qquad \vdots \\ \{t\} \times \ldots\ldots\ldots \times \{t\} \end{matrix}}_{\delta}$$

Let

$$S = (\mathscr{F}_1(S_0) \otimes \ldots \otimes \mathscr{F}_1(S_{\delta-1})) \cdot T$$

which is defined as follows

$$S = \{f_0(a_0) \otimes \ldots \otimes f_{\delta-1}(a_{\delta-1}) \mid f_i \in \mathscr{F}_1(S_i),\ a_i \in A^*,\ i < \delta\} \cup$$

$$\cup \{f_0(a_0) \otimes \ldots \otimes f_{\delta-2}(a_{\delta-2}) \otimes f_{\delta-1}(t) \mid f_i \in \mathscr{F}_1(S_i),$$

$$a_i \in A^*,\ i < \delta\} \cup$$

$$\cup \{f_0(a_0) \otimes \ldots \otimes f_{\delta-3}(a_{\delta-3}) \otimes f_{\delta-2}(t) \otimes f_{\delta-1}(t) \mid f_i \in \mathscr{F}_1(S_i),$$

$$a_i \in A^*,\ i < \delta\} \cup$$

$$\vdots$$

$$\cup \{f_0(t) \otimes \ldots \otimes f_{\delta-1}(t) \mid f_i \in \mathscr{F}_1(S_i),\ i < \delta\}.$$

A typical element of S consists of δ blocks, where the i-th block is obtained by substituting a letter from A into $f_i \in \mathscr{F}_1(S_i)$. Moreover if the i-th block is obtained by substituting the letter t then in all following blocks the letter t is substituted. As the parameter words $f_i \in \mathscr{F}_1(S_i) \subseteq [A]\binom{n_i}{1}$ are defined on the smaller alphabet A^* and do not contain the letter t, the only way that the letter t occurs in an element of S is by substitution of t for the parameter in a parameter word f_i. In order to show that $S \xrightarrow{A} (1)_\delta$ we check conditions (α), (β).

For (α): In order to show that $\mathscr{F}_2(S) = \phi$ assume that there exists $h \in \mathscr{F}_2(S)$. Decompose h into its blocks, i.e. $h = h_0 \otimes h_1 \otimes \ldots \otimes h_{\delta-1}$, where the length of the block h_i is n_i $(i < \delta)$.

Fact 1. No block is a 2-parameter word. For otherwise it would be a 2-parameter word over the alphabet A^*, contradicting the choice of S_i.

Fact 2. No block contains simultaneously a parameter and the letter t. For otherwise such a block is a 2-parameter word over the alphabet A^* in which t acts as a parameter, contradicting the choice of S_i.

Fact 3. Facts 1, 2 imply that $h(t, a) \notin S$ for all $a \in A^*$. For otherwise $\min h^{-1}(\lambda_0) < \min h^{-1}(\lambda_1)$ implies that $h(t, a)$ contains a block without letter t following blocks containing t. This contradicts the definition of S via the scheme T.

These facts show that $\mathcal{F}_2(S) = \phi$.

For (β): Let $\Delta: S \to \delta$ be a coloring. Consider first the coloring

$$\Delta_{\delta-1}: S_{\delta-1} \to \delta_{\delta-1} = \delta^{|S_{\delta-2} \times \ldots \times S_0|}$$

defined by

$$\Delta_{\delta-1}(g) = (\Delta(g_0 \otimes \ldots \otimes g_{\delta-2} \otimes g) \mid g_i \in S_i,\ i < \delta - 1),$$

where for convenience $\Delta(g_0 \otimes \ldots \otimes g_{\delta-2} \otimes g)$ shall be 0 if $g_0 \otimes \ldots \otimes g_{\delta-2} \otimes g$ does not belong to S.

By choice of $S_{\delta-1}$ there exists $f_{\delta-1} \in \mathcal{F}_1(S_{\delta-1})$ monochromatic for $\Delta_{\delta-1}$. Consider next the coloring

$$\Delta_{\delta-2}: S_{\delta-2} \to \delta_{\delta-2} = \delta^{2|S_{\delta-3} \times \ldots \times S_0|}$$

defined by

$$\Delta_{\delta-2}(g) = (\Delta(g_0 \otimes \ldots \otimes g_{\delta-3} \otimes g \otimes f_{\delta-1}(\alpha)) \mid g_i \in S_i \text{ for } i < \delta - 2 \text{ and } \alpha \in A).$$

By choice of $f_{\delta-1}$:

$$\Delta(g_0 \otimes \ldots \otimes g_{\delta-3} \otimes g \otimes f_{\delta-1}(\alpha)) = \Delta(g_0 \otimes \ldots \otimes g_{\delta-3} \otimes g \otimes f_{\delta-1}(\beta))$$

for all $\alpha, \beta \in A^*$. Thus the only relevant distinction whether $\alpha \in A^*$ or $\alpha = t$ contributes a factor 2 in the exponent of the number of colors of $\Delta_{\delta-2}$. By choice of $S_{\delta-2}$ there exists $f_{\delta-2} \in \mathcal{F}_1(S_{\delta-2})$ monochromatic for $\Delta_{\delta-2}$. Proceed iteratively and consider for $i = 1, \ldots, \delta$

$$\Delta_{\delta-i}: S_{\delta-i} \to \delta_{\delta-i} = \delta^{i|S_{\delta-i-1} \times \ldots \times S_0|}$$

defined by

$$\Delta_{\delta-i}(g) = (\Delta(g_0 \otimes \ldots \otimes g_{\delta-i-1} \otimes g \otimes f_{\delta-i+1}(\alpha_1) \otimes \ldots$$
$$\ldots \otimes f_{\delta-i+j}(\alpha_j) \otimes f_{\delta-i+j+1}(t) \otimes \ldots \otimes f_{\delta-1}(t)) \mid$$
$$g_0 \in S_0, \ldots, g_{\delta-i-1} \in S_{\delta-i-1}, \ \alpha_1 \ldots \alpha_j \in A^*, \ 0 \leqslant j < i).$$

Observe again that in the definition of $\Delta_{\delta-1}$ the particular choice of the letters α is not relevant, i.e. for $\alpha_1 \ldots \alpha_j, \beta_1 \ldots \beta_j \in A^*$

$$\Delta(g_0 \otimes \ldots \otimes g_{\delta-i-1} \otimes g \otimes f_{\delta-i+1}(\alpha_1) \otimes \ldots$$
$$\ldots \otimes f_{\delta-i+j}(\alpha_j) \otimes f_{\delta-i+j+1}(t) \otimes \ldots \otimes f_{\delta-1}(t)) =$$
$$= \Delta(g_0 \otimes \ldots \otimes g_{\delta-i-1} \otimes g \otimes f_{\delta-i+1}(\beta_1) \otimes \ldots$$
$$\ldots \otimes f_{\delta-i+j}(\beta_j) \otimes f_{\delta-i+j+1}(t) \oplus \ldots \oplus f_{\delta-1}(t)).$$

By choice of $S_{\delta-i}$ find $f_{\delta-i} \in \mathscr{F}_1(S_{\delta-i})$ monochromatic for $\Delta_{\delta-i}$.

Finally consider the δ-parameter word

$$f = f_0 \otimes \ldots \otimes f_{\delta-1} \in \mathscr{F}_1(S_0) \otimes \ldots \otimes \mathscr{F}_1(S_{\delta-1}).$$

The induced coloring Δ_f acts constantly on the rows of scheme T. Thus the $\delta + 1$ many rows are colored with δ colors. Two of them have the same color, thus defining a 1-parameter word which is monochromatic for Δ.∎

4.

An immediate corollary of Hales–Jewett's theorem is a partition theorem for affine points:

Theorem 4.1 [3]. *Let* $\mathbf{F}$ *be a finite field and let* δ, m *be positive integers. Then there exists a positive integer* n *such that for every coloring of the affine points in the* n*-dimensional affine space over* $\mathbf{F}$ *with* δ*-many colors, i.e. for every coloring* $\Delta\colon \mathbf{F}^n \to \delta$, *there exists a monochromatic* m*-dimensional affine subspace.*

Luckily enough the configuration S which has been constructed in the proof of the Main theorem also yields the following restricted version of the above theorem, viz.

Theorem 4.2. *Let* $\mathbf{F}$ *be a finite field and let* δ, m *be positive integers. Then there exist a positive integer* n *and a set* $S \subseteq \mathbf{F}^n$ *of affine points such that*

(α) S *does not contain an* $(m + 1)$*-dimensional affine space,*

(β) *For every coloring of the points of* S *with* δ *many colors there exists an affine* m*-dimensional monochromatic subspace which is contained in* S.

Proof. Again the crucial case is $m = 1$, which may be proved as follows: Let $\mathbf{F} = \{0, 1, \ldots, t\}$ be the finite field, where 0 is the zero element of $\mathbf{F}$. Let $S \subseteq \mathbf{F}^n = [\mathbf{F}]\binom{n}{0}$ be the set constructed in the proof of Lemma 3.3.

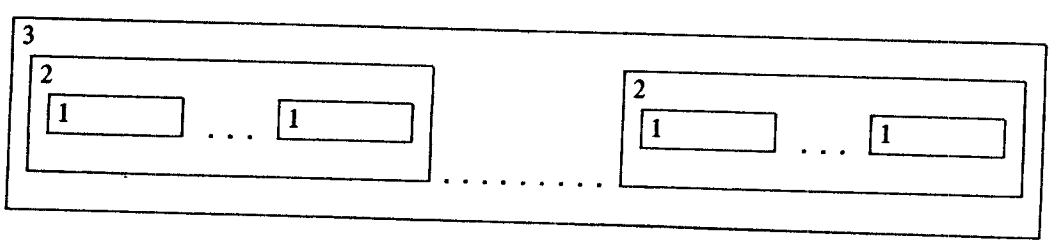
We claim that S has the desired properties. As in particular each $f \in [\mathbf{F}]\binom{n}{m}$ represents an m-dimensional affine subspace of $\mathbf{F}^n$ (compare e.g. [5]) property (β) follows immediately. In order to show (α) we have to be more careful because there exist certain m-dimensional affine subspaces which are not represented by any $f \in [\mathbf{F}]\binom{n}{m}$.

As S is defined recursively we may think of members of S being arranged in blocks of blocks of ... of blocks. There are t "levels" of blocks. The blocks of the first level are always of the form

$$0 \ldots 0\ a \ldots a\ 1 \ldots 1,$$

where $a \in \mathbf{F}$.

The following diagram depicts the structure of S for GF (4), where the small numbers indicate the levels of the corresponding blocks:



Let $L = \{\bar{c} + \alpha\bar{x} \mid \alpha \in \mathbf{F}\}$ be an affine line and assume that L is contained in S.

We examine the implications of the assumption "$L \subseteq S$" on the vectors $\bar{c}$ and $\bar{x}$. In particular we look at the structure of $\bar{c}$ and $\bar{x}$ in the blocks of level 1.

Claim. Let B be a block of level 1. Then the parts of $\bar{c}$ and $\bar{x}$ belonging to this block B – which will be denoted by $\bar{c}_B$, resp. $\bar{x}_B$ – are of the form

$$\bar{c}_B = 0 \ldots 0\ b \ldots b\ 1 \ldots 1$$

$$\bar{x}_B = 0 \ldots 0\ x \ldots x\ 0 \ldots 0,$$

where $b, x \in \mathbf{F}$.

Proof of claim. Let B be any block of level 1 and assume that $\bar{x}_B$ is not the zero vector. Let $\bar{c}_B = c_1 \ldots c_n$ and $\bar{x}_B = x_1 \ldots x_n$.

1. Assume that $x_i \neq 0$ and $c_j = 0$, where $i < j$, then $x_i = x_j$. For otherwise consider $\bar{c} + \left(\frac{1}{x_i}\right)\bar{x}$ and find a block of level 1 with an entry 1 preceding an entry different from 1, again contradicting the structure of blocks of level 1.

2. Assume that $x_i \neq 0$, $c_i = 0$ and $c_j \neq 0$ for some $j > i$, then $x_j = 0$. For otherwise consider $\bar{c} - \left(\frac{c_j}{x_j}\right)\bar{x}$ and find a block of level 1 with a non-zero entry preceding a zero entry, again a contradiction.

3. Assume that $c_i = 0$ and $c_j \neq 0$, $c_j \neq 1$ for some $i < j$, then $x_i = 0$. For otherwise 1. and 2. imply that

$$\bar{c}_B = 0 \ldots 0\ 0 \ldots 0\ a \ldots a\ 1 \ldots 1,$$

where $a \neq 0, 1$ and

$$\bar{x}_B = 0 \ldots 0\ x \ldots x\ 0 \ldots 0\ 0 \ldots 0.$$

Consider $\bar{c} + \left(\frac{1}{x}\right)\bar{x}$ and find a block of level 1 with an entry 1 preceding an entry different from 1, a contradiction.

Now the claim may be proved as follows: let i be the minimal index such that $x_i \neq 0$ and let $\alpha \in \mathbf{F}$ be such that for $\bar{d} = \bar{c} + \alpha\bar{x}$, where $\bar{d}_B = d_1 \ldots d_n$, it follows that $d_i = 0$. By 1., 2. and 3. above then $\bar{d}_B$ and $\bar{x}_B$ have the following structure:

$$\bar{d}_B = 0 \ldots 0 \; 0 \ldots 0 \; 1 \ldots 1$$

$$\bar{x}_B = 0 \ldots 0 \; x \ldots x \; 0 \ldots 0,$$

but then $\bar{c} = \bar{d} - \alpha\bar{x}$ and thus

$$\bar{c}_B = 0 \ldots 0 \; b \ldots b \; 1 \ldots 1$$

$$\bar{x}_B = 0 \ldots 0 \; x \ldots x \; 0 \ldots 0,$$

where $b = -\alpha x$ gives the desired result.∎

Let $P = \{\bar{c} + \alpha\bar{x} + \beta\bar{y} \mid \alpha, \beta \in \mathbf{F}\}$ be a configuration of points which is contained in S. We show that $\bar{x}$ and $\bar{y}$ are linearly dependent, thus S does not contain a 2-dimensional affine subspace.

As an immediate corollary from the claim one obtains that for every block B of level 1 the vectors $\bar{c}_B, \bar{x}_B$ and $\bar{y}_B$ have the following structure:

$$\bar{c}_B = 0 \ldots 0 \; b \ldots b \; 1 \ldots 1$$

$$\bar{x}_B = 0 \ldots 0 \; x \ldots x \; 0 \ldots 0$$

$$\bar{y}_B = 0 \ldots 0 \; y \ldots y \; 0 \ldots 0.$$

The proof of Theorem 4.2 will be finished by showing that for every $k = 1, \ldots, t$ and every block B of level k the following holds:

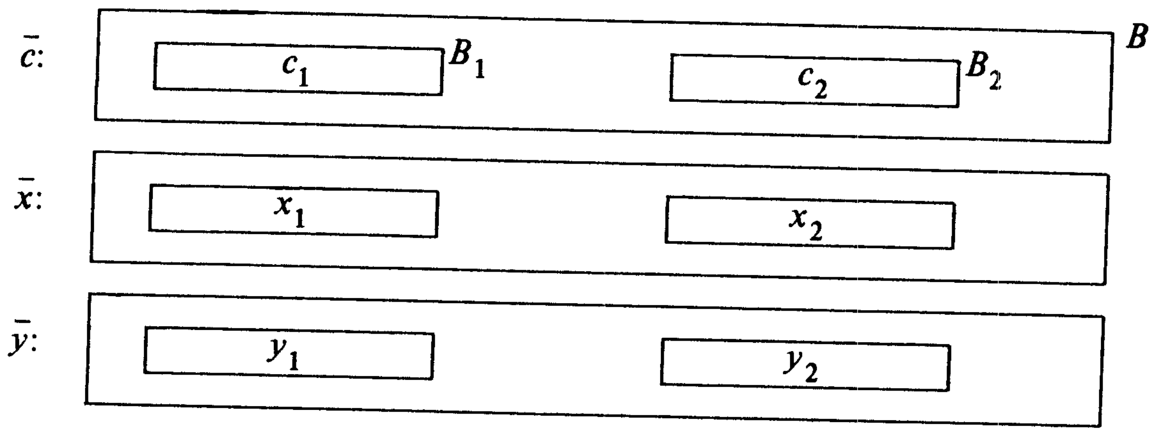
(*) $\bar{x}_B$ and $\bar{y}_B$ are linearly dependent

(**) if $k < t$ and if for some $\alpha, \alpha', \beta, \beta' \in \mathbf{F}$ both $(\bar{c} + \alpha\bar{x} + \beta\bar{y})_B$ and $(\bar{c} + \alpha'\bar{x} + \beta'\bar{y})_B$ contain at least one entry of t then all entries of t in $(\bar{c} + \alpha\bar{x} + \beta\bar{y})_B$ and $(\bar{c} + \alpha'\bar{x} + \beta'\bar{y})_B$ occur exactly at the same positions.

The assertions (*) and (**) are proved by induction on k. The case $k = 1$ follows from the claim. So assume that (*) and (**) hold for $k - 1$

and consider a block B of level $k > 1$. Suppose $(*)$ does not hold on B. Then there must be two $(k-1)$-level blocks B_1, B_2 in B on the entries of the union of which $\bar{x}$ and $\bar{y}$ are linearly independent.

By induction $\bar{x}_{B_1}$ and $\bar{y}_{B_1}$ are linearly dependent and $\bar{x}_{B_2}$ and $\bar{y}_{B_2}$ are linearly dependent. Let c_1, x_1, y_1 be the entries in the first position of B_1 where not both x_1, y_1 are equal to zero and let c_2, x_2, y_2 be similarly defined for B_2. The situation is as follows:



Then $x_1 y_2 \neq x_2 y_1$ by choice of B_1 and B_2 and the induction assumption on B_1, B_2. Thus we can find α, β so that $c_1 + \alpha x_1 + \beta y_1 = t$ but $c_2 + \alpha x_2 + \beta y_2 \neq t$. From property $(**)$ we see that $(\bar{c} + \alpha\bar{x} + \beta\bar{y})_{B_2}$ does not contain any t. This violates the conditions for S. Hence $(*)$ holds on B.

Now let $k < t$. We show that $(**)$ holds for B. Assume that it does not. By $(*)$ we need only consider vectors of form $\bar{c} + \alpha\bar{x}$ on B. Then for some α, α' we must have $(\bar{c} + \alpha\bar{x})_B$ and $(\bar{c} + \alpha'\bar{x})_B$ with different entries of t. In particular there exists a $(k-1)$-level block B_1 in B such that $(\bar{c} + \alpha\bar{x})_{B_1}$ and $(\bar{c} + \alpha'\bar{x})_{B_1}$ have different entries of t. By induction on $(**)$ then one of these blocks, say $(\bar{c} + \alpha\bar{x})_{B_1}$, contains some entries of t while $(\bar{c} + \alpha'\bar{x})_{B_1}$ does not contain any t. In particular there exists another $(k-1)$-level block B_2 such that $(\bar{c} + \alpha'\bar{x})_{B_2}$ contains some entries of t. By the rules for S the block B_1 precedes the block B_2 and thus also $(\bar{c} + \alpha\bar{x})_{B_2}$ contains some entries of t. Again by induction on $(**)$ then all entries of t in $(\bar{c}+\alpha\bar{x})_{B_2}$ and $(\bar{c}+\alpha'\bar{x})_{B_2}$ occur exactly at the same positions. This means that for a suitable α'', the vector $\bar{c}+\alpha''\bar{x}$ has entries of t in B_2 and entries of $t-1$ in B_1. This violates the rules for S, which exclude any entry of $t-1$ preceding an entry of t in any block of level $t-1$ or smaller. Thus $(**)$ holds. The induction is now complete and letting $k=1$ we see that $(*)$ implies that $P=\{\bar{c}+\alpha\bar{x}+\beta\bar{y} \mid \alpha, \beta \in \mathbf{F}\}$ is just a line at best and not a plane. ∎

Now much is known about restricted versions of Graham and Rothschild's partition theorem for k-parameter words [3] with $k>0$. The only result in this direction is due to Nešetřil and Rödl [7] who announced the case of 2-parameter words over the empty alphabet:

Theorem 4.1 [7]. *Let* m, δ *be positive integers. There exist a positive integer* n *and a subset* S *of* $[\phi]\binom{n}{2}$ *not containing an* $(m+1)$*-dimensional subcube (i.e. for all* $g \in [\phi]\binom{n}{m+1}$ *there exists an* $h \in [\phi]\binom{m+1}{2}$ *with* $g \cdot h \notin S$*) such that for every coloring* $\Delta\colon S \to \delta$ *there exists an* $f \in [\phi]\binom{n}{m}$ *which is contained* S *and monochromatic with respect to* Δ.

Even for the empty alphabet the general case remains unsettled. Also nothing is known about restricted versions of the partition theorems for finite vector spaces [2].

Finally it could be worthwhile to note that in case of colorings of 0-parameter words (i.e. $k=0$) and requiring a monochromatic 1-parameter word not only a restricted but also simultaneously a restricted and induced version may be established, viz.

Theorem 4.3. *Let* A *be a finite set and let* $I \subseteq A$ *be a subset of* A. *Then for every positive integer* δ *there exists a positive integer* n *and a set* $S \subseteq [A]\binom{n}{0}$ *such that*

(α) S *does not contain a 2-parameter word,*

(β) *for every coloring* $\Delta\colon S \to \delta$ *there exists a 1-parameter word*

$f \in [A]\binom{n}{1}$ *such that*

(i) *all elements* $f \cdot a$, $a \in I$, *are colored the same, and thus particularly* $f \cdot a \in S$ *for* $a \in I$;

(ii) $f \cdot a \notin S$ *for* $a \notin I$.

This strengthens a result of [9], where an induced version of Hales–Jewett's theorem has been established. Obviously the same strengthening applies to Theorem 4.2 as well. We do not know whether in general a restricted and simultaneously induced version of Hales–Jewett's theorem is valid.

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