Review of[1]

Slicing the Truth:
On the Computability Theoretic
and Reverse Mathematical Analysis
of Combinatorial Principles
by Denis Hirschfeldt
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1 Introduction

When teaching Discrete Math I may ask the students the following:

From the theorem that every number factors uniquely into primes prove that $\sqrt{2}$ is irrational.

A student submitted the following:

1. Every number factors uniquely into primes.
2. It is well known the if p is prime then $\sqrt{p}$ is irrational.
3. 2 is a prime.
4. Hence $\sqrt{2}$ is irrational.

There are two things wrong with the above proof for what I intended to ask: (1) it never uses that $\sqrt{2}$ is irrational, (2) the basic assumptions that it uses are to strong.

Episodes like the one sketched above are very rare. The class does have the (correct) sense that when I say *use* A *to prove* B I mean that the proof should (1) use A, and (2) only use easy math steps.

The program of reverse mathematics formalizes this notion and tries to unify all of mathematics into equivalent theorems. One goal is to examine which theorems *require* nonconstructive proofs and, in a sense, how nonconstructive. We give one example. Let WKL be the weak Konig's lemma: every infinite binary tree has an infinite branch. The following are equivalent: (1) WKL, (2) $[0, 1]$ is compact (henceforth $COMPACT$).

2 Summary of Contents

The first few chapters of the book discuss the reverse mathematics program due to Steve Simpson and Harvey Friedman. There is a base-system of axioms called RCA_0 and all equivalences are proven there. For example, one can show $WKL \implies COMPACT$ and $COMPACT \implies WKL$ with all reasoning in RCA_0. From a proof theory prospective RCA_0 is weak, though much of mathematics can be done in it, including elementary number theory and most theorems in finite combinatorics. There are four other systems which form a hierarchy: WKL_0 (RCA_0 plus the weak

Konig's lemma), ACA_0, ATR_0, and Π^1_1-CA_0. The R in RCA_0 stands for recursive (computable)–all of the objects you can show exist are computable. WKL_0 lets you do a few other things than what is computable; however, there is a model of WKL_0 where all of the sets are low. The A in ACA_0 stands for Arithmetic. The set of all arithmetic sets is a model for ACA_0. Virtually all of mathematics can be done in ACA_0. ATR_0 and Π^1_1 are not discussed much. The five classes are called *the big five*.

The author gives some examples of theorems in math that fit exactly into one of these five classes and points us to Steve Simpson's book *Subsystems of Second Order Arithmetic* where even more theorems are so classified. It is hard to measure how many or what percent of theorems fit exactly into one of the big five; however, enough do to make the classification interesting.

However, the author goes in an entirely different direction. Ramsey Theory for pairs *is not* equivalent to *any* of these classes. Ramsey for triples and beyond is equivalent to ACA_0.

Notation:

- RT^n_c is Ramsey theory for n-tuples and c colors.
- RT^∞_c is $(\forall n)[RT^n_c]$.
- RT^n_∞ is $(\forall c)[RT^n_c]$.
- RT is $(\forall c)(\forall n)[RT^n_c]$.

Chapter 6 is the real heart of the book. In this chapter they classify many RT^n_c in terms of the big five.

1. For all $n \geq 2$ there is a computable 2-coloring of $\binom{\mathsf{N}}{n}$ with no Σ_n homogenous set. Hence for all $n \geq 1$, RT^n_2 is not equivalent to RCA_0 (all we needed was that there was no computable homogenous set). The same holds for c-colorings if $c \geq 2$.

2. The usual proofs of Ramsey's theorem show that for all $n \geq 1$, $c \geq 2$, RT^n_c can be proven in ACA_0. This does not preclude the possibility of being provable in a lower system.

3. Every computable 2-coloring of $\binom{\mathsf{N}}{2}$ has a low_2 homogenous set. This is the key ingredient in the proof that RT^2_2 does not imply ACA_0. (The result that RT^2_2 does not imply ACA_0 was first proven by Seetapun; however, the proof in this book using low_2 sets is a newer easier proof.)

4. There exists a computable 2-coloring of $\binom{\mathsf{N}}{3}$ such that every homogenous set computes HALT. Together with point 2 this implies (with some work) that ACA_0 and RT^3_2 are equivalent. This extends to RT^n_c for all $c \geq 2$ and $n \geq 3$.

5. RT is not in ACA_0 but it is in an extension called ACA'_0.

Chapter 7 is about theories being conservative. For example, if ϕ is a sentence in the language of PA then $PA \vdash \phi$ iff $ACA_0 \vdash \phi$. Chapter 8 has nice diagrams summarizing the results in Chapter 6.

Chapter 9 is about weaker versions of Ramsey Theory (there was also some of this in Chapter 6) and how they relate to each other and to RCA_0 It would have been helpful to have a list of all of the variants of Ramsey Theory; hence I provide one here.

1. RT^n_c: For all c-colorings of $\binom{\mathsf{N}}{n}$ there is a homogenous set.
2. $RT^n_{<\infty c}$: $(\forall c)[RT^n_c]$.
3. $RT^{<\infty}_c$: $(\forall n)[RT^n_c]$.
4. RT : $(\forall c)(\forall n)[RT^n_c]$.
5. SRT^2_c: For all stable c-colorings of $\binom{\mathsf{N}}{2}$ there is a homogenous set. A stable coloring $COL : \binom{\mathsf{N}}{2} \rightarrow [c]$ is one such that, for all x, $\lim_{y\rightarrow\infty} COL(x,y)$ exists.
6. COH : For all countable sequences of sets of naturals $R_1, R_2, R_3, \ldots$ there exists an infinite set C (called a *cohesive set*) such that, for all i, $C \subseteq^* R_i$ or $R_i \subseteq^* C$. This follows from RT^2_2.
7. ADS : Every infinite linear orderings has either an infinite ascending subsequence or an infinite descending subsequence.
8. $SADS$: Every Stable infinite linear orderings has either an infinite ascending subsequence or an infinite descending subsequence. An ordering is *stable* if it is discrete and every element has either a finite number of elements less than it or greater than it. A nontrivial example is $\omega + \omega^*$ where ω^* is the naturals in reverse order.
9. $CADS$: Every infinite linear orderings has a stable suborder.
10. CAC : Every infinite partial order has either an infinite Chain or an infinite Anti-Chain.
11. $SCAC$: Every Stable infinite partial order has either an infinite Chain or an infinite Anti-Chain.
12. $CCAC$: Every infinite Stable partial order has an infinite Stable suborder.
13. EM (Erdos-Moser): If T is a tournament on N then there is an infinite $A \subseteq \mathsf{N}$ on which T is transitive. A tournament is a directed graph where, for all x, y, exactly one of $R(x,y)$ or $R(y,x)$ holds.
14. $FS(n)$ (Free Set): For all $f : \binom{\mathsf{N}}{n} \rightarrow \mathsf{N}$ there is an infinite $A \subseteq \mathsf{N}$ such that for all $s \in \binom{A}{n}$ either $f(s) \in A$ or $f(s) \in s$.
15. $TS(n)$ (Thin Set): For all $f : \binom{\mathsf{N}}{n} \rightarrow \mathsf{N}$ there is an infinite $A \subseteq \mathsf{N}$ such that $f(\binom{A}{n}) \neq \mathsf{N}$.
16. FIP (Finite Intersection Principle): Every nontrivial family of sets has a maximal subfamily with the finite intersection property. A family of sets satisfies *finite intersection property* if every finite subfamily has a nonempty intersection.

Chapter 10 is about theorems that are beyond ACA_0. We give two examples:

1. A *well partial order (wpo)* is a partial order that has neither infinite descending sequences or infinite antichains. J. Kruskal showed that the set of trees under embedability (or under minor) form a wpo. We denote this KTT. H. Friedeman showed that $ATR_0 \not\vdash KTT$. Hence KTT is a natural theorem which requires a rather strong proof system.
2. Laver showed that the set of all countable linear orderings under embedding is a wpo. This is called FRA since it was original Fraisee's conjecture. Shore showed that FRA implies ATR_0, hence it also requires a rather strong proof system.

3 Opinion of the Book

Who can read this book? To read this book you need to already know some computability theory and some Ramsey theory. Knowing some Reverse math would also be good; however, that is less necessary. Many theorems are left for the exercises so the reader has to do some work themselves.

Who should read this book? The book gathers together in one place most of the theorems known about where Ramsey Theory and some variants of it fit into the Reverse math framework. The book also discusses many combinatorial principles that the reader may not realize are really Ramsey Theory, but they are!

There are two theorems that I was surprised were not discussed. (1) Mileti has done work on the reverse mathematics of *the Canonical Ramsey Theorem* that does not seem to have been discussed, and (2) Schmerl has done work on the reverse mathematics of the chromatic number of a graph.

However, if you care about the proof strength of Ramsey Theory, this is THE book for you!

4 Opinion of the Field

(Keep in mind that THIS section really is just MY opinion.)

When is asking where theorems fit into the Reverse Math framework interesting?

1) When it leads to new proofs of old theorems. Jockusch's proof that every computable coloring of $\binom{\mathsf{N}}{2}$ has a Π_2-homogenous set can be presented as a different proof of Ramsey Theory without even mentioning Π_2, but noting that the proof is vaguely more constructive.

2) When it leads to interesting computability theory. The construction of a computable coloring that has no Σ_2 homogenous sets is interesting.

3) When you tie together many different theorems as being equivalent. This is similar to in NP-completeness you need to think of SAT and HAM CYCLE as being *the same problem.*

But I do have one criticism. One of the main results in this book is that RT^2_2 is definitely weaker in proof strength than RT^3_2. But the usual proofs of RT^2_2 and RT^3_2 really don't seem any different from a constructive point of view. AH-HA- hence there should be a new proof of RT^2_2 that is more constructive. Or more something. Alas, I've asked people in the field and they can't really point me to this better proof.

Added later: I've discussed this point with Denis Hirschfeldt when I gave him a copy of the review and we may soon have a different proof of Ramsey Theory inspired by these results.