Joint Review of[1]

The Mathematics of Various Entertaining Subjects:
Research in Recreational Math
and
The Mathematics of Various Entertaining Subjects Vol 2:
Research in Games, Graphs, Counting, and Complexity
and
The Mathematics of Various Entertaining Subjects Vol 3:
The Magic of Mathematics

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1 Introduction

Each of these three books is a collection of articles on recreational math. Or are they? What is recreational math? The following three points are neither necessary or sufficient for an article to be on recreational mathematics; however, it is a good guide.

1. The problem being discussed is accessible to a layperson. So the topic is often Combinatorics or Number Theory, but never Algebraic Geometry or Topos.

2. The answer to the problem can be understood by a layperson.

3. This one is optional: A layperson has a chance to solve the problem themselves.

4. I've used the term layperson without defining it. And I won't.

Ray Smullyan wrote the forward to Volume 1 which is appropriate since, as is often quoted:

Ray Smullyan worked in both recreational and serious mathematics and made a mockery of the distinction between the two.

In this review I will describe a few articles from each book, and discuss how recreational it was. In light of the quote above, I will not try to hard to distinguish "recreational math" from "serious math".

Why study recreational math?

1. Sometimes it leads so serious math, as this book proves.

2. Sometimes it serves as the starting point for more serious math.
3. Sometimes it can be used to get youngsters interested in math. Many people of my generation (I claim I am 62 though at one time Wikipedia thought I was 10 years older than I am) read Martin Gardner's recreational mathematics column in Scientific American and were inspired by them.
4. BECAUSE IT IS FUN!

2 Vol 1: Research in Recreational Mathematics

2.1 Heartless Poker by Dominic Lanphier and Laura Taalman

No, heartless poker is not poker played without mercy (I am not sure what that means). It refers to playing poker with a deck that has no hearts. This chapter is not really about that. Its about what happens if you vary (a) the number of ranks, (b) the number of suites, (c) the size of a hand.

1. In standard Poker (13 ranks, 4 suits, 5-card hand) a straight (5 cards in increasing rank order, no wrap-around) is more likely than a flush (all he same suite) which is more likely than a full house (a pair and 3-of-a-kind). But the probabilities are close together (Straight ~ 0.00392, Flush ~ 0.00196, Full House ~ 0.00144.) This is recreational.
2. By changing the number of ranks and the number of suits can you get all six possibly orderings of what is more likely? You can! This is recreational(?).
3. Can you get all three probabilities to be the same. No. This is recreational(?).
4. Can you get any two of the probabilities to be the same. No. This requires using some hard number theory as a black box. This is not recreational(?).

2.2 Analysis of Crossword Puzzle Difficulty Using a Random Graph Process by John K. McSweeney

Given a crossword puzzle, how hard is it to solve? This article *is not* about complexity theory, $n \times n$ grids, and NP-completeness. This article *is* about actual crossword puzzles from the newspaper that you do over breakfast. Or if its hard then over breakfast, lunch, and dinner.

There are two factors that make a crossword puzzle hard.

1. The difficulty of the clues. And if it has several words intersecting it then the difficulty of the clue given partial information. They model this by giving, for each clue, a threshold that tells you what fraction of the letters need to be there in order to solve it (recreational). Later they use probability distributions for thresholds (perhaps not recreational).
2. The structure. They model this with a bipartite graph where the ACROSS words are on the left, the DOWN words are on the right, and if they intersect there is an edge between them.

The article looks at real examples and also which properties of the bipartite graph make a puzzle easier or harder. They then use their techniques on *real examples.*

This article is easy to read if you skim some of the random process stuff. Nevertheless, reading it is easier than doing the Sunday New York Times crossword puzzle.

2.3 The Cookie Monster Problem by Leigh Marie Braswell and Tanya Khovanova

There are k jars of cookies. Jar 1 has c_1 cookies, ..., Jar k has c_k cookies. A Cookie monster wants to eat all of the cookies. He will do this in a sequence of moves. On move j he will pick a number a_j and a set of jars $I \subseteq \{1, \ldots, k\}$ and eat a_j cookies from the jars in I. (The jars in I must all have $\geq a_j$ cookies.)

Here is an example: The jars have (1,5,9,10) cookies. On move 1 he eats 9 cookies from the 3rd and 4th jar, so now the jars are (1,5,0,1). On move 2 he eats 1 cookie from the 1st, 2nd, and 4th jar, so now the jars are (0,4,0,0). On move 3 he eats 4 cookie from the 2nd jar, so now the jars are (0,0,0,0). That took 3 moves. Can he do better? I leave that as an exercise.

More generally, given a set of jars, what is the *minimum* number of moves needed to eat all of the cookies?

In this chapter they give many strategies and sequences of jars and prove several theorems. They are all pleasant and easy. Definitely recreational. The Fibonacci numbers make a surprising appearance. Actually its not surprising: this section of the book is titled *Fibonacci Numbers*. That aside, why are the Fibonacci numbers in this chapter? Because they given an example of a sequence of jars that take a large number of moves.

3 Vol 2: Research in Games, Graphs, Counting, and Complexity

3.1 The Cycle Prisoners by Peter Winkler

There are n people in prison. They do not know n. They do have a leader L. The warden will do the following: Every night the prisoners write down a bit on a piece of paper. The warden collects them, looks at them, and then redistributes them to the prisoners via a cyclic shift. This goes on for a while until the prisoners all announce that they all know the n. At that point, they all go free! A few details: (a) the leader can broadcast instructions to them before the game begins, and (b) no communication between the prisoners once the game starts.

There are two startling facts about this game: (1) there is a solution, and (2) it has applications to distributed computing. I do not know which one is more startling.

The game is easy to describe (I just did). The solution is very very clever but understandable. Recreational! Though I will point out I doubt the layperson, or even a seasoned professional, can come up with the answer.

3.2 Duels, Truels, Gruels, and the Survival of the Unfittest by Dominic Lanphier

A *Duel* is as follows: (a) there are two players Alice and Bob who have guns, (b) Alice has probability p_A of hitting Bob, Bob has probability p_B of hitting Alice, (c) they shoot in order Alice, Bob, Alice, Bob, ... until there is only one person alive, (d) one shot that hits will kill.

The questions one asks are: (1) What is the probability that Alice survives? (2) What is the probability that Bob survives? (3) What is the expected number of rounds?

Duels has been well studied and are completely understood. What makes it easy is that (in contrast to Truels and n-ruels) neither player has a decision to make: Alice will aim at Bob, and Bob will aim at Alice.

What if you have three players?

A *Truel* is as follows: (a) there are three players Alice, Bob, Carol who have guns, (b) Alice has probability p_A of hitting whoever she aims at, Bob has probability p_B of hitting whoever he aims at, Carol has probability p_C of hitting whoever she aims at, (c) they shoot in order Alice, Bob, Carol, Alice, Bob, Carol, ... until there is only one person alive; whoever shoots can *choose* who they try to kill, (d) one shot that hits will kill.

Truels have been studied but are not fully understood. What makes it hard is that (in contrast to Duels) each player must *decide* who to aim at. An interesting aspect is that there are times the weakest shooter has the highest probability of surviving. For more on Truel's see Brams & Kilgour [2] and Kilgour [3, 4, 5].

This article looks at three variants of these concepts.

1. n-uels: There are n people.
2. Gruels: There are n people and they are on the vertices of a graph. Alice can aim to kill Bob only if there is an edge from Alice to Bob. (The graph is undirected.)
3. A person needs more than one shot to kill them.

This article covers a lot of old and new ground. The math is mostly easy: elementary algebra and probability. Some recurrences and some generating functions. Mostly recreational and perhaps a good introduction to recurrences and generating functions.

4 Vol 3: The Magic of Mathematics

4.1 Probability in Your Head by Peter Winklers

Eight problems in probability are presented which the author claims can be done in your head. All of the problems are (1) easy to understand and, (2) have solutions that are easy to understand. Indeed— I did not even need pencil and paper when reading the solutions. Are they problems that a layperson could do on their own? Hard to say.

Here is one of the problems (I won't give the answer):

Three points are chosen at random on a circle. What is the probability that there is a semicircle of that circle containing all three.

In the answer section he gives the trick to being able to do it in your head and then states the (then easily proven) result for n points.

4.2 Losing Checkers is Hard by Jefferey Bosboom, Spencer Congero, Erik D. Demaine, Martin L. Demaine, and Jason Lynch

The article talks about winning and losing checkers on a given $n \times n$ board which need not be the usual starting configuration. Robson [6] showed that determining who wins is EXPTIME-complete. The article explains what this means intuitively.

The article proves the following results:

1. Given a board and pieces on it, and its RED's turn, can RED make a move so that BLACK is forced to win in one move? This problem is PSPACE-complete. You might want to lose at checkers if playing a child, a Wookie, or a Wookie Child. See

`https://www.starwars.com/video/let-the-wookiee-win`

2. Consider the variant where if a player cannot capture a piece then they lose (this would not be interesting if the game has the standard initial placement of pieces). Determining if a given player has win is EXPTIME-complete.

3. Consider the variant where the players are cooperating and their goal is to eliminate all of the pieces of one color. The problem of determining if they can do this in 2 moves is NP-complete.

The problems are understandable to the layperson, if they can accept on faith that NP-complete, PSPACE-complete, and EXPTIME-complete are a hierarchy of hard problems. The proofs would be hard for a layperson but understandable and interesting for the readers of this review.

4.3 On Partitions into Squares of Distinct Integers Whose Reciprocals Sum to 1 by Max Alekseyev

Graham [1] proved the following:
For all $n \geq 78$ *there exists* $x_1 < \cdots < x_k \in \mathbb{N}$ *such that (a)* $\sum_{i=1}^{k} x_i = n$*, and (b)* $\sum_{i=1}^{k} \frac{1}{x_i} = 1$*. (77 is tight*

He conjectured that there was some N such that, for all $n \geq N$, his theorem with $\sum_{i=1}^{k} x_i$ replaced with $\sum_{i=1}^{k} x_i^2 = n$ holds.

This paper proves the conjecture. In particular they show the following:
For all $n \geq 8543$ *there exists* $x_1 < \cdots < x_k \in \mathbb{N}$ *such that (a)* $\sum_{i=1}^{k} x_i^2 = n$*, and (b)* $\sum_{i=1}^{k} \frac{1}{x_i} = 1$.

In this article they prove both Graham's theorem and Graham's conjecture. They do this in a way that is very motivated. They never use the term *induction* so that the layperson can follow it.

The result is a nice combination of math and CS.

5 Opinion

Anyone reading this review will enjoy most of the articles in these books. They are a perfect storm of (a) problems of interest, (b) solutions of interest, (c) new to the reader, and (d) understandable to the reader.

But what about people who are not as mathematically mature as readers of this column? This question arises since the books advertise themselves as *Recreational Mathematics* which means that laypeople should also benefit. Looking over all of the articles with this in mind we have an almost-perfect storm. They all hit a,b, and c. And they are all well written. Some of them fail on point d. Nevertheless, exposure is still good. For example, knowing that losing at checkers is a hard problem is still valuable, even if you don't quite know what that means, or the proof that its true.

References

[1] R. Graham. A theorem on partitions. *Journal of the Australian Math Society*, 3, 1963. `http://www.math.ucsd.edu/~ronspubs/`.

[2] S. Grams and D. M. Kilgour. The Truel. *Mathematics Magazine*, 70:315–326, 1997.

[3] D. M. Kilgour. The simultaneous Truel. *International Journal of Game Theory*, 1:229–242, 1972.

[4] D. M. Kilgour. The sequential Truel. *International Journal of Game Theory*, 4:151–172, 1974.

[5] D. M. Kilgour. Equilbrium points of infinite sequential Truels. *International Journal of Game Theory*, 6:167–180, 1977.

[6] J. M. Robson. N by N checkers is Exptime complete. *SIAM Journal on Computing*, 13:252–267, 1984.
`https://epubs.siam.org/doi/10.1137/0213018`.