Problems with a Point: Exploring Math and Computer Science

April 11, 2020

Authors:
William Gasarch
Clyde Kruskal

April 11, 2020

How This Book Came to Be

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Book's Origin

- In 2003 Lance Fortnow started **Complexity Blog**
- In 2007 Bill Gasarch joined and it was a co-blog.
- In 2015 various book publishers asked us

  Can you make a book out of your blog?

- Lance declined but Bill said **YES**.

Book's Point

Bill took the posts that had the following format:

- make a point **about** mathematics
- do some math to underscore those points

and made those into chapters.

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Caveat: Not every chapter is quite like that.
To quote Ralph Waldo Emerson

A foolish consistency is the hobgoblin of small minds.

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The publisher wisely decided to be less cute and more informative:
Problems with a Point: Exploring Math and Computer Science

Clyde Joins the Project!

After some samples of Bill's writing the publisher said

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Please Procure People to Polish Prose and Proofs of Problems with a Point

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Now onto some samples of the book!

Point: Students Can Give Strange Answers

April 11, 2020

The Paint Can Problem

From the Year 2000 Maryland Math Competition:
There are 2000 cans of paint. Show that at least one of the following two statements is true:

- There are at least 45 cans of the same color.
- There are at least 45 cans that are different colors.

Work on it.

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Work on it.
Answer:
If there are 45 different colors of paint then we are done. Assume there are ≤ 44 different colors. If all colors appear ≤ 44 times then there are $44 \times 44 = 1936 < 2000$ cans of paint, a contradiction.
Note: this was Problem 1, which is supposed to be easy and indeed 95% got it right. What about the other 5%? Next slide.

One of the Wrong Answers. Or is it?

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ANSWER:

Paint cans are grey. Hence there are all the same color. Therefore there are 2000 cans that are the same color.

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What do you think:

- Thats just stupid. 0 points.
- Question says *cans of the same color*. . .. The full 30 pts.
- Not only does he get 30 points, but everyone else should get 0.

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ANSWER:

If you look at a paint color really really carefully there will be differences. Hence, even if two cans seem to both be (say) RED, they are really different. Therefore there are 2000 cans of different colors.

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What do you think:

- Thats just stupid. 0 points.
- Well. . . he's got a point. 30 points in fact.
- Not only does he get 30 points, but everyone else should get 0.

A Triangle Problem

From the year 2007 Maryland Math Competition.

QUESTION: Let ABC be a fixed triangle. Let COL be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.

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Note I think I was assigned to grade it since it looks like the kind of problem I would make up, even though I didn't. It was problem 5 (out of 5) and was hard. About 100 students tried it, 8 got full credit, 10 got partial credit

Funny Answers One

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Funny Answer One:
All the vertices are red because I can make them whatever color I want. I can also write at a 30 degree angle to the bottom of this paper (The students answer was written at a 30 degree angle to the bottom of the paper.) if thats what I feel like doing at the moment. Just like $2 + 2 = 5$ *if thats what my math teacher says. Math is pretty subjective anyway.*

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Theorem The students is not serious.
Proof Assume, by contradiction, that they are serious. Then they really think math is subjective. Hence they don't really understand math. Hence they would not have done well enough on Part I to qualify for Part II. But they took Part II. Contradiction.

Funny Answers Two

QUESTION: Let ABC be a fixed triangle. Let COL be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.

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Was Student Two Serious. Yes. About **Justice!**.

The Real Answer to Points in the Plane Problem

Each point in the plane is colored either red or green. Let ABC be a fixed triangle. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.

Fix a 2-coloring of the plane.

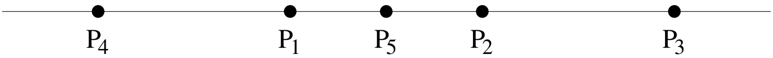
There are 3 equally-spaced mono points on x-axis

Proof Clearly there are two points on the x-axis of the same color: p_1, p_2 are RED. If p_3, the midpoint of p_1, p_2, is RED then p_1, p_3, p_2 are all RED. DONE. Hence we assume p_3 is GREEN.

Let p_4 be such that $|p_1 - p_4| = |p_2 - p_1|$. If p_4 is RED then p_4, p_1, p_2 are all RED. DONE. Hence we assume p_4 is GREEN.

Let p_5 be such that $|p_5 - p_2| = |p_2 - p_1|$. If p_5 is RED then p_1, p_2, p_5 are all RED. DONE. Hence we assume p_5 is GREEN.

Only case left p_3, p_4, p_5 are all GREEN. DONE.



Finish Proof By Picture

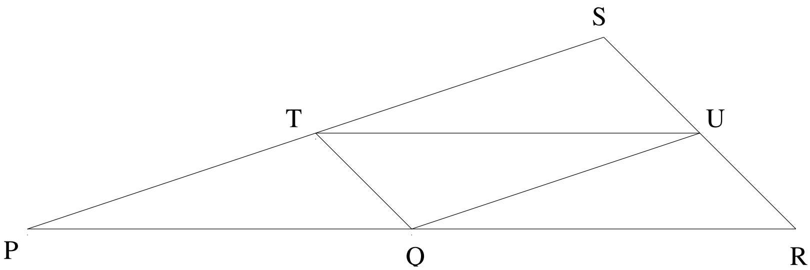


Figure: Triangle Similar to ABC with Monochromatic Vertices

P, Q, R are RED.

If T or U or S are RED then get RED Triangle similar to ABC.

If not then ALL of T, U, S are GREEN, so get GREEN triangle similar to ABC.

Point: What is a Pattern?

April 11, 2020

Simple Functions

Bill assigned the following in Discrete Math: For each of the following sequences find a **simple function** $A(n)$ such that the sequence is $A(1), A(2), A(3), \ldots$

1. 10, -17, 24, -31, 38, -45, 52, $\cdots$
2. -1, 1, 5, 13, 29, 61, 125, $\cdots$
3. 6, 9, 14, 21, 30, 41, 54, $\cdots$

Caveat: These are NOT trick questions.
Work on it.

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1. 10, -17, 24, -31, 38, -45, 52, $\cdots$ $A(n) = (-1)^{n+1}(7n+3)$.
2. -1, 1, 5, 13, 29, 61, 125, $\cdots$ $A(n) = 2^n - 3$.
3. 6, 9, 14, 21, 30, 41, 54, $\cdots$ $A(n) = n^2 + 5$.

A Student asks — What is a Simple Function?

One student, in earnest, emailed Bill the following:

We never defined **Simple Function** *in class so I went to Wikipedia. It said that* **A Simple Function is a linear combination of indicator functions of measurable sets.** *Is that what you want us to use?*

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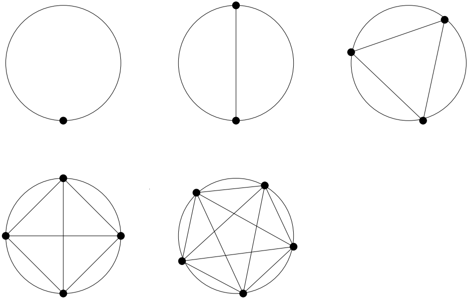
I told him NO— all I wanted is an easy-to-describe function. I should have told him to use that definition to see what he came up with.
The student got the first one right, but left the other two blank.

When Do Patterns Hold?

The last question brings up the question of when patterns do and don't hold. We looked for cases where a pattern *did not* hold.

First Non-Pattern: n Points on a circle

What is the max number of regions formed by connecting every pair of n points on a circle. For $n = 1, 2, 3, 4, 5$:



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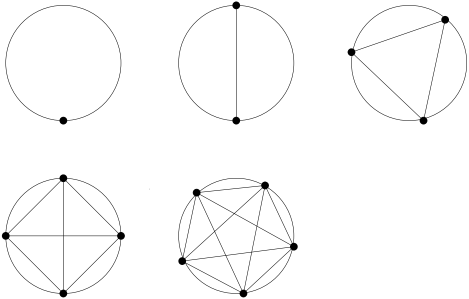

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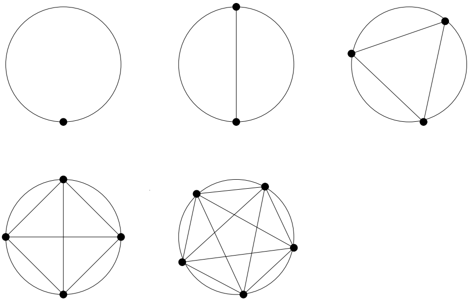

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But for $n = 6$, the number of regions is only 31.
The actual number of regions for n points is $\binom{n}{4} + \binom{n}{2} + 1$.

Second Non-Pattern: Borwein Integrals

$$\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin x}{x} \frac{\sin \frac{x}{3}}{\frac{x}{3}} = \frac{\pi}{2}$$

$$\vdots$$

$$\int_0^\infty \frac{\sin x}{x} \frac{\sin \frac{x}{3}}{\frac{x}{3}} \frac{\sin \frac{x}{5}}{\frac{x}{5}} \frac{\sin \frac{x}{7}}{\frac{x}{7}} \frac{\sin \frac{x}{9}}{\frac{x}{9}} \frac{\sin \frac{x}{11}}{\frac{x}{11}} \frac{\sin \frac{x}{13}}{\frac{x}{13}} = \frac{\pi}{2}$$

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But

$$\int_0^\infty \frac{\sin x}{x} \frac{\sin \frac{x}{3}}{\frac{x}{3}} \frac{\sin \frac{x}{5}}{\frac{x}{5}} \frac{\sin \frac{x}{7}}{\frac{x}{7}} \frac{\sin \frac{x}{9}}{\frac{x}{9}} \frac{\sin \frac{x}{11}}{\frac{x}{11}} \frac{\sin \frac{x}{13}}{\frac{x}{13}} \frac{\sin \frac{x}{15}}{\frac{x}{15}} =$$

$$\frac{467807924713440738696537864469\pi}{935615849440640907310521750000} \sim 0.99999999998529371186 \times \frac{\pi}{2}$$

Why the breakdown at 15?

Because

$$\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{13} < 1$$

but

$$\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{15} > 1.$$

For more Google

Borwein Integral

Computers to FIND proofs vs Computers to DO Proofs

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Colorings and Square Differences

The following are all true:

1. There exists a number W_2 such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

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4. For all c there exists a number W_c ....

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The proofs in the literature of these theorems give EEEEEEEEEENORMOUS bounds on W_2, W_3, W_4, W_c. We look at easier proofs with two **points** in mind:

- Would they make good questions on a HS math competition.
- The role of Computers in these proofs.

2-colorings and Square Differences

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Since 1 is a square $\mathrm{COL}(2) = B$.

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AH-HA: $\mathrm{COL}(1) = \mathrm{COL}(5)$ and $5 - 1 = 4 = 2^2$. So $W_2 \leq 5$.

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AH-HA: $RBRB$ shows that $W_2 \leq 5$.

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Since 1 is a square $\mathrm{COL}(3) = R$.
Since 1 is a square $\mathrm{COL}(4) = B$.
Since 1 is a square $\mathrm{COL}(5) = R$.
AH-HA: $\mathrm{COL}(1) = \mathrm{COL}(5)$ and $5 - 1 = 4 = 2^2$. So $W_2 \leq 5$.
AH-HA: $RBRB$ shows that $W_2 \leq 5$.
So $W_2 = 4$.

2-colorings and Square Differences

There exists a number W_2 such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

Think About how to prove it and what W_2 is.

Let COL be a 2-coloring of $\{1, 2, 3, \ldots\}$ with colorings R and B.
We can assume $\mathrm{COL}(1) = R$.
Since 1 is a square $\mathrm{COL}(2) = B$.
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Upshot Could be easy HS Math Comp Prob. No computer used.

3-colorings and Square Differences

In W_2-proof had $\mathrm{COL}(1) = \mathrm{COL}(5)$. Need similar for W_3.
Let COL be 3-coloring of $\{1, 2, 3, \ldots\}$, uses R, B, G.
$\mathrm{COL}(1) = R$.

3-colorings and Square Differences

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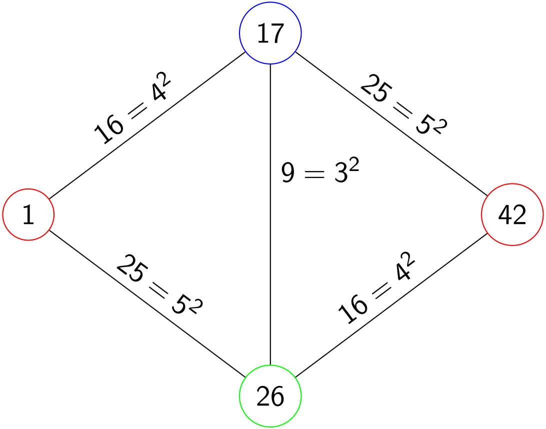

Figure: $\mathrm{COL}(x) = \mathrm{COL}(x + 41)$

Since $\mathrm{COL}(x) = \mathrm{COL}(x + 41) \ldots$

Use $\mathrm{COL}(x) = \mathrm{COL}(x + 41)$ to finish the proof and find upper bound on W_3.

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Think about this

$$\mathrm{COL}(1) = \mathrm{COL}(1{+}41) = \mathrm{COL}(1{+}2{\times}41) = \cdots = \mathrm{COL}(1{+}41{\times}41)$$

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$W_3 \leq 1 + 41^2 = 1682$

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So 1 and 41^2 are a square apart and the same color.
$W_3 \leq 1 + 41^2 = 1682$
Can we get better bound on W_3?

Better Bound on W_3

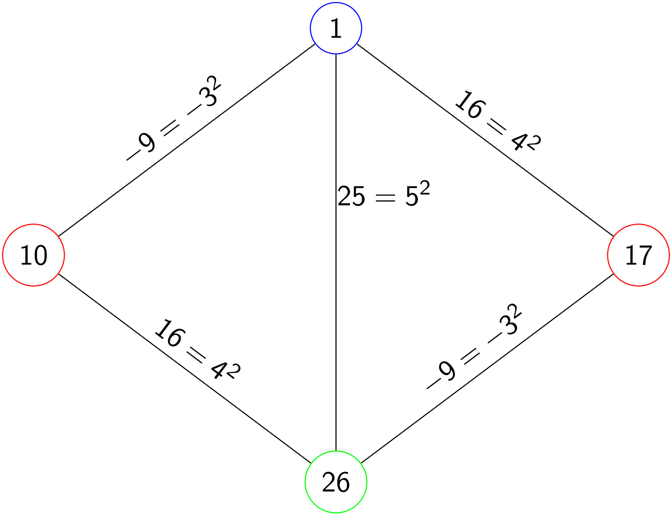

Figure: If $x \geq 10$ then $\mathrm{COL}(x) = \mathrm{COL}(x+7)$, so $W_3 \leq 59$

Reflection on W_3

1. Problem 5 (so hard) on UMCP HS Math Comp, 2006:
   Show that for all 3-colorings of $\{1, \ldots, 2006\}$ *there exists 2 numbers that are a square apart that are the same color*
2. 240 took exam, 40 tried this problem, 10 got it right.
3. Bill Gasarch and Matt Jordan proved, by hand, $W_3 = 29$.
4. **Is there a HS-proof that W_4 exists?** Bill wanted to put this problem on the next HS exam to find out. He was (wisely) told **NO**.
5. The question still remains: Is there a HS proof that W_4 exists?

Reflection on W_3

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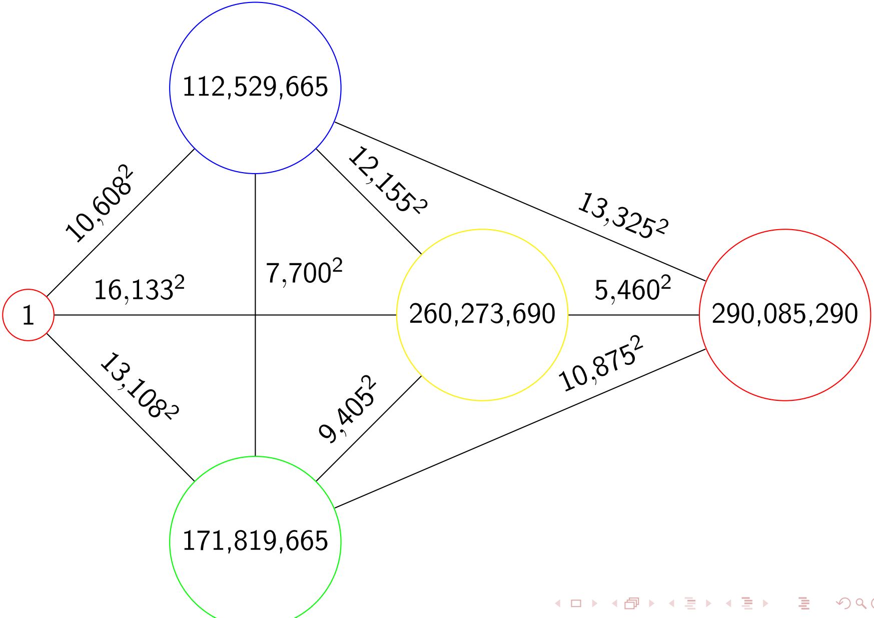
W_4 Exists: $\mathrm{COL}(x) = \mathrm{COL}(x + 290{,}085{,}290)$



Reflection on W_4

1. Zach's proof shows $W_4 \leq 1 + 299,085,290^2$.
   PRO Proof is easy to verify
   CON Number is large, proof does not generalize to W_5.
2. The classical proof.
   PRO Gives bounds for W_c.
   CON Bounds are GINORMOUS, even for W_2.
3. A Computer Search showed that $W_4 = 58$.
   PRO Get exact value.
   CON not human-verifiable. Does not generalize to W_5.

Which do you prefer?

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