Funky Dice: An Exposition

William Gasarch - University of MD

If you roll two standard 6-sided dice then

1. 2: (1,1). ONE way. Prob $\frac{1}{36}$.
2. 3: (1,2), (2,1). TWO ways. Prob $\frac{1}{18}$.
3. 4: (1,3), (2,2), (3,1). THREE ways. Prob $\frac{1}{12}$.
4. 5: (1,4), (2,3), (3,2), (4,1). FOUR ways. Prob $\frac{1}{9}$.
5. 6: (1,5), (2,4), (3,3), (4,2), (5,1) FIVE ways. Prob $\frac{5}{36}$.
6. 7: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) SIX ways. Prob $\frac{1}{6}$.
7. 8: (2,6), (3,5), (4,4), (5,3), (6,2) FIVE ways. Prob $\frac{5}{36}$.
8. 9: (3,6), (4,5), (5,4), (6,3) FOUR ways. Prob $\frac{1}{9}$.
9. 10: (4,6), (5,5), (6,4) THREE ways. Prob $\frac{1}{12}$.
10. 11: (5,6), (6,5) TWO ways. Prob $\frac{1}{18}$.
11. 12: (6,6) ONE way. Prob $\frac{1}{36}$.

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2. Can you label the dice something other than $\{1, \ldots, 6\}$ and $\{1, \ldots, 6\}$ and get the same probabilities you get with standard dice?

Loaded Dice

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Fair Dice Yield Unfair Sums

Fair Die:

$$\mathbf{Pr(1)=Pr(2)=Pr(3)=Pr(4)=Pr(5)=Pr(6) = 1/6 \sim 0.167}$$

Roll TWO of them.

Pr(Sum=2)=1/36 (This is Min Pr(Sum))
Pr(Sum=7)=1/6. (This is Max Pr(Sum))

Sums are Unfair!

How Unfair?: 1/6 - 1/36 $\sim$ 0.139 unfair.

What Are Loaded Dice?

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1. Does there exist a pair of loaded dice such that the sums all have equal probability $1/11$?
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3. NO, no such dice can exist! (We prove on next few slides.)

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More generally, the coefficient of x^i is $\mathrm{Prob}(\mathrm{sum} = \mathrm{i})$.

No Dice!

Let $(p_1, \ldots, p_6)$ and $(q_1, \ldots, q_6)$ be dice. **Assume** they yield **fair sums**, all sums have prob $1/11$. Then

$$(p_6x^6 + \cdots + p_1x^1)(q_6x^6 + \cdots + q_1x^1) = \frac{1}{11}(x^{12} + x^{11} + \cdots + x^2)$$

So

$$(p_6x^5 + \cdots + p_1)(q_6x^5 + \cdots + q_1) = \frac{1}{11}(x^{10} + x^9 + \cdots + x + 1)$$

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Continued on Next Slide.

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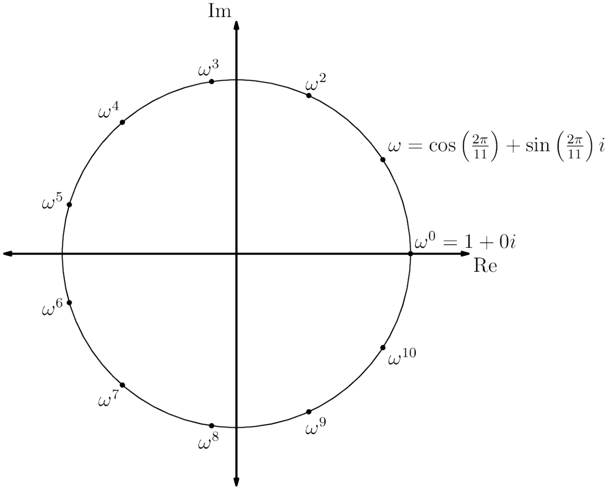
The roots of $x^{11} - 1$ are on the complex unit circle. See Next Slide.

The 11th Roots of Unity: Only Real one is 1



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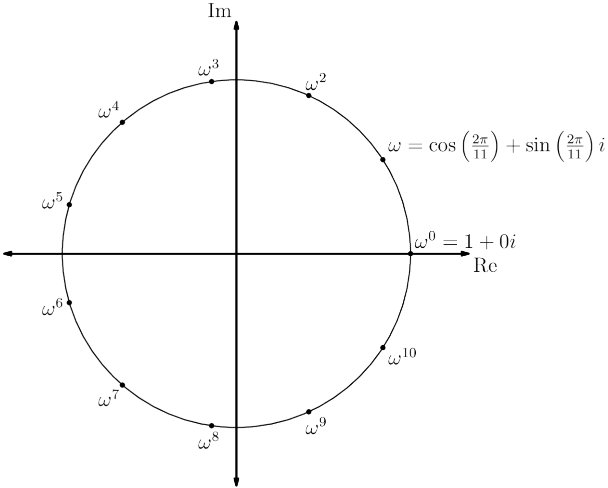

1 is only real 11th root of unity. $x^{10} + \cdots + 1 = 0$: **no** real roots.

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Recap

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Contradiction

What About Two d-Sided Dice?

For which $d \geq 2$ can you load two d-sided dice to get fair sums?
VOTE:

1. No d.
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1. The proof that for even d you **cannot** load two d-sided dice to get fair sums is similar to what we did for two 6-sided dice.
2. The proof that for odd d you **cannot** load two d-sided dice to get fair sums requires new techniques.

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Fame! One paper refers to **The Gasarch-Kruskal Theorem**.

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How far are normal dice from uniform?

$$2(1/11-1/36)^2+2(1/11-1/18)^2+2(1/11-1/12)^2+2(1/9-1/11)^2+$$

$$2(5/36-1/11)^2)+(1/6-1/11)^2 \sim 0.0217$$

How Close To Uniform Can You Get? (cont)

Theorem The optimal pair of 6-sided dice is $(\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2})$ and $(\frac{1}{8}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{8})$.

How Close To Uniform Can You Get? (cont)

Theorem The optimal pair of 6-sided dice is
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Distance from Uniform is $\frac{1}{352} \sim 0.0028$.

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One measure is distance from uniform which is what Asgarli, Hartclass, Ostrov, Walden used.

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1. **Normal Dice** They were $1/6$ - $1/36 \sim 0.139$ unfair.
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What About n-sided Dice?

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The optimal pair of n-sided dice is
$(\frac{1}{2}, 0, \ldots, 0, \frac{1}{2})$

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The distance from uniform is $\frac{1}{2(2n-1)(3n-2)}$.

Different Labels on Dice

William Gasarch - University of MD

Can You Label Dice To Get Same Probs?

A **labeling** of a 6-sided die has any positive natural numbers as labels. We allow using a number twice. We allow using numbers higher than 6. So $(1, 2, 2, 3, 5, 8)$ would be allowed.

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YES. We prove this.

Let Polynomials Do The Work For You!

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Coefficient of x^n is number of ways to get n.

Example of Non-Standard Labelings

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1. 12: TWO ways. Prob $\frac{1}{18}$.
2. 9: THREE ways. Prob $\frac{1}{12}$.
3. 8: NINE ways. Prob $\frac{1}{4}$.
4. 6: TWO ways. Prob $\frac{1}{18}$.
5. 5: TWELVE ways. Prob $\frac{1}{3}$.
6. 4: FOUR ways. Prob $\frac{1}{9}$.
7. 3: THREE ways. Prob $\frac{1}{12}$.
8. 2: ONE ways. Prob $\frac{1}{36}$.

Is there a Non-Standard Labeling That...

Question Is there a nonstandard labeling of two 6-sided dice that gives the same probabilities as the standard dice?

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$$(x^6+x^5+x^4+x^3+x^2+x)^2.$$

Is there a Non-Standard Labeling That... Cont.

$$(x^{a_1}+x^{a_2}+x^{a_3}+x^{a_4}+x^{a_5}+x^{a_6})(x^{b_1}+x^{b_2}+x^{b_3}+x^{b_4}+x^{b_5}+x^{b_6}) =$$

$$(x^6+x^5+x^4+x^3+x^2+x)^2 = x^2(x^5+x^4+x^3+x^2+x+1)^2 =$$

$$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2.$$

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$x(x+1)(x^2-x+1)^2(x^2+x+1) = x^8 + x^6 + x^5 + x^4 + x^3 + x.$
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So desired dice are $(1,2,2,3,3,4)$ and $(1,3,4,5,6,8)$.

What About Two d-Sided Dice?

For which $d \ge 2$ are there two non-standard d-sided dice that have the same prob as standard dice? **VOTE:**

1. All even d.
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1. The proof is similar to what we did, though requires some thought.

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For which $d_1, d_2 \geq 2$ are there non-standard d_1-sided die and d_2-sided die that have the same prob as standard dice? **VOTE:**

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3. Gallian and Rusin's paper exactly characterizes when this is possible: `https://www.cs.umd.edu/~gasarch/BLOGPAPERS/nonstandarddice.pdf`
   The paper only looked at n d-sided dice. I do not know if someone else later did the case of d_1, d_2.

Parting Thoughts

William Gasarch - University of MD

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2. Polynomials are useful for problems with dice since multiplication gives information.
3. It is remarkable that a problem about dice lead to looking at complex roots of polynomials!
4. Congratulations for doing well on the UMCP HS Math Competition!