How big & how small can the set of all subset-sums be for a set of size n.

1. $\{2^0, 2^1, \ldots, 2^{n-1}\}$ has all 2^n subset-sums. This is of course max.

2. $\{1, \ldots, n\}$ has $\frac{n(n+1)}{2} + 1$ subset-sums. Lance proved that this is min.

Are there any other subsets of $\{1, \ldots, n\}$ that have $\frac{n(n+1)}{2}+1$ subset-sums?
Stupid Answer Yes, take $\{x, 2x, \ldots, nx\}$.
$(\exists A)[A \neq \{x, \ldots, nx\}]$ that has $\frac{n(n+1}{2}$ subset-sums?
$n = 3$: $\{a, b, a+b\}$. sums $\{0, a, b, a+b, 2a+b, a+2b, 2a+2b\}$. 7 of them.
For $n \geq 4$ we will show that the answer is NO.

Suppose a set $A = \{a_1 < \cdots < a_n\}$ has $\frac{n(n+1)}{2} + 1$ subsetsums. Now add a larger number b to the set. Suppose the new set has $\frac{(n+1)(n+2)}{2} + 1$ subsetsums.

$\frac{n(n+1)}{2} + 1$ of them are all the subsetsums of the A with b added to them. So, we can have at most $n + 1$ other subsetsums that do not contain b. Since the subsets $\emptyset$ $\{a_1\}$,$\{a_2\}$,$\{a_3\}$,... $\{a_n\}$ are $n + 1$ subsetsums which are less than b, there must be no other subsetsums. This implies that b is the smallest number greater than a_n which is a subsetsum of A. This also implies that if $A \cup \{b\}$ has minimal subsetsums, then so does A. So, we only need to find all the 4-element sets that have 11 subsetsums.

We know that any candidate set will look like $\{a, b, a + b, 2a + b\}$ where $a < b$

That set will have the following 11 subsetsums:

0
a
b
$a + b$
$2a + b$
$3a + b$
$2a + 2b$
$3a + 2b$
$4a + 2b$
$3a + 3b$
$4a + 3b$
as well as $a + 2b$

So, $a + 2b$ must equal one of the above numbers. The only possibility is $3a + b$. So, $a + 2b = 3a + b$, which implies $b = 2a$. So, our set had to be $\{a, 2a, 3a, 4a\}$.