1 Limits On How Well We Can Approximate $a + \sqrt{b}$ With Rationals

Let SQ be the set of squares of naturals. Let SQQ be the set of squares of rationals.

We want to find see how well we can approximation $a + \sqrt{b}$. Since shifting by a natural number does not change how well the $a + \sqrt{b}$ can be approximated, and we want $\sqrt{b}$ to be real and irrational, we only look for $a \in \{0\} \cup \mathsf{Q} - \mathsf{N}$ and $b \in \mathsf{Q}^+ - \mathsf{SQQ}$.

We get conditions on a, b and the approximation bounds at the end.

We will determine a, b, Δ to satisfy the following:

$$(\exists^\infty p, q \in \mathsf{N})\left[\left|\frac{p}{q} - (a + \sqrt{b})\right| < \frac{\Delta}{q^2}\right] \implies \text{A CONTRADICTION.}$$

Assume p, q, Δ are such that $\left|\frac{p}{q} - (a + \sqrt{b})\right| < \frac{\Delta}{q^2}$.

We will find (a, b, Δ) such that if q is large we get a contradiction.

There exists $\delta < \Delta$ such that

$$\left|\frac{p}{q} - (a + \sqrt{b})\right| = \frac{\delta}{q^2}.$$

$$p - q(a + \sqrt{b}) = \frac{\delta}{q}$$

$$\frac{\delta}{q} = p - aq - \sqrt{b}q$$

$$\frac{\delta}{q} + \sqrt{b}q = p - aq$$

$$\left(\frac{\delta}{q} + \sqrt{b}q\right)^2 = (p - aq)^2$$

$$\frac{\delta^2}{q^2} + 2\frac{\delta}{q}\sqrt{b}q + q^2 b = p^2 - 2apq + q^2a^2$$

$$\frac{\delta^2}{q^2} + 2\delta\sqrt{b} = p^2 - 2apq + q^2a^2 - q^2b$$

$$\frac{\delta^2}{q^2} + 2\delta\sqrt{b} = p^2 - 2apq + q^2a^2 - q^2b = p^2 - 2apq + q^2(a^2 - b)$$

Want that as $q \to \infty$ LHS $\notin \mathsf{Z}$ and RHS $\in \mathsf{Z}$.
LHS $\notin \mathsf{Z}$: $2\delta\sqrt{b} < 1$, so $\delta < \frac{1}{2\sqrt{b}}$.
SO we can take $\Delta = \frac{1}{2\sqrt{b}}$.
RHS $\in \mathsf{Z}$.

Note that p, q could be ANYTHING IN Z. Hence we need to make have $2apq \in \mathsf{Z}$ and $a^2 - b \in \mathsf{Z}$. Recall that $a \in \{0\} \cup \mathsf{Q} - \mathsf{N}$ and $b \in \mathsf{Q}^+ - \mathsf{SQQ}$.

To make $2apq \in \mathsf{Z}$ need $a \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$.

1. $a = 0$: To make $q^2(a^2 - b) \in \mathsf{Z}$ we need $b \in \mathsf{N} - \mathsf{SQ}$.

   Upshot It works to take $a = 0$, $b \in \mathsf{N} - \mathsf{SQ}$, and $\Delta = \frac{1}{2\sqrt{b}}$.

2. $a = \frac{1}{2}$: To make $q^2(a^2 - b) \in \mathsf{Z}$ we need $q^2(\frac{1}{4} - b) \in \mathsf{Z}$. Hence

$$b \in X = \left\{ \frac{c}{4} : c \equiv 1 \pmod 4, c \notin \mathsf{SQ} \right\}.$$

   Upshot It works to take $a = \frac{1}{2}$, $b = \frac{c}{4}$ where $c \equiv 1 \pmod 4$, $c \notin \mathsf{SQ}$, and $\Delta = \frac{1}{2\sqrt{b}} = \frac{1}{\sqrt{c}}$.

3. $a = -\frac{1}{2}$: To make $q^2(a^2 - b) \in \mathsf{Z}$ we need $q^2(\frac{1}{4} - b) \in \mathsf{Z}$. Hence

$$b \in Y = \left\{ \frac{c}{4} : c \equiv 1 \pmod 4, c \notin \mathsf{SQ} \right\}.$$

   Upshot It works to take $a = -\frac{1}{2}$, $b = \frac{c}{4}$, $c \equiv 1 \pmod 4$, $c \notin \mathsf{SQ}$, and $\Delta = \frac{1}{2\sqrt{b}} = \frac{1}{\sqrt{c}}$.

How big does q have to be?
Need

$$\frac{\delta^2}{q^2} + 2\delta\sqrt{b} < 1$$

δ is at most Δ, so we need

$$\Delta^2 + 2\Delta\sqrt{b}q^2 < q^2$$

$$\Delta^2 + 2\Delta\sqrt{b}q^2 < q^2$$

$$\Delta^2 < q^2(1 - 2\Delta\sqrt{b})$$

$$q^2 > \frac{\Delta^2}{1 - 2\Delta\sqrt{b}}$$

a	b	$\Delta = \frac{1}{2\sqrt{b}}$	empirical Δ
0	2	$\frac{1}{2\sqrt{2}} =$	
0	3	$\frac{1}{2\sqrt{3}} =$	
0	5	$\frac{1}{2\sqrt{5}} =$	
$\frac{1}{2}$	$\frac{5}{4}$	$\frac{1}{\sqrt{5}} =$	
0	6	$\frac{1}{2\sqrt{6}} =$	
0	7	$\frac{1}{2\sqrt{7}} =$	
0	8	$\frac{1}{2\sqrt{8}} =$	
0	10	$\frac{1}{2\sqrt{10}} =$	
0	11	$\frac{1}{2\sqrt{11}} =$	
0	12	$\frac{1}{2\sqrt{12}} =$	
0	13	$\frac{1}{2\sqrt{13}} =$	
$\frac{1}{2}$	$\frac{13}{4}$	$\frac{1}{\sqrt{13}} =$	