Find the Missing Numbers

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Alice recites **ALL BUT** $k$ of the numbers in $\{1, \ldots, n\}$ in some random order.

We denote these numbers $x_1, \ldots, x_{n-k}$.

Missing numbers are $y_1, \ldots, y_k$.

Bob wants to find $y_1, \ldots, y_k$ but only has $O(k \log n)$ space.
$k = 1$ Case

Bob computes $\sum_{i=1}^{n-1} x_i$ and finds

$$y_1 = \sum_{i=1}^{n} i - \sum_{i=1}^{n-1} x_i = \frac{n(n+1)}{2} - \sum_{i=1}^{n-1} x_i.$$
Bob computes $\sum_{i=1}^{n-2} x_i$ and $\sum_{i=1}^{n-2} x_i^2$.

$$y_1 + y_2 = \frac{n(n + 1)}{2} - \sum_{i=1}^{n-2} x_i$$

$$y_1^2 + y_2^2 = \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n-2} x_i^2 = \frac{n(n + 1)(2n + 1)}{6} - \sum_{i=1}^{n-2} x_i^2.$$  

WANT $y_1y_2$ (you’ll see why soon)

$$y_1y_2 = \frac{(y_1 + y_2)^2 - (y_1^2 + y_2^2)}{2}.$$
Clearly

\[ X^2 - (y_1 + y_2)X + y_1 y_2 \]

\textbf{KEY STEP:} Form Poly

\[ k = 2 \text{ Case Continued} \]
KEY STEP: Form Poly

\[ X^2 - (y_1 + y_2)X + y_1y_2 \]

\[ = (X - y_1)(X - y_2) \]

Find its roots. THEY ARE THE MISSING NUMBERS!!!!
$k = 3$ Case- The Main Idea

**NEED**

\[ y_1 + y_2 + y_3 \]

\[ y_1y_2 + y_1y_3 + y_2y_3 \]

\[ y_1y_2y_3 \]

Form polynomial

\[ X^3 - (y_1 + y_2 + y_3)X^2 + (y_1y_2 + y_1y_3 + y_2y_3)X - y_1y_2y_3 \]

\[ = (X - y_1)(X - y_2)(X - y_3) \]

Find its roots. **THEY ARE THE MISSING NUMBERS!**
Solution One

Bob computes $\sum_{i=1}^{n-3} x_i$ and $\sum_{i=1}^{n-3} x_i^2$ and $\sum_{i=1}^{n-3} x_i^3$.

$$y_1 + y_2 + y_3 = \frac{n(n+1)}{2} - \sum_{i=1}^{n-2} x_i$$

$$y_1^2 + y_2^2 + y_3^2 = \frac{n(n+1)(2n+1)}{6} - \sum_{i=1}^{n-2} x_i^2$$

$$y_1^3 + y_2^3 + y_3^3 = \sum_{i=1}^{n} i^3 - \sum_{i=1}^{n-2} x_i^3 = \frac{n^2(n+1)^2}{4} - \sum_{i=1}^{n-2} x_i^3$$

From these CAN get

$$y_1 + y_2 + y_3, \quad y_1y_2 + y_1y_3 + y_2y_3, \quad y_1y_2y_3$$

Messy!- On Next Slides
Deriving Sym Functions From Sums of Powers

Have

\[ y_1 + y_2 + y_3, \quad y_1^2 + y_2^2 + y_3^2, \quad y_1^3 + y_2^3 + y_3^3 \]

Want

\[ y_1 + y_2 + y_3 \text{ have}, \quad y_1y_2 + y_1y_3 + y_2y_3, \quad y_1y_2y_3 \]

Get \( y_1y_2 + y_1y_3 + y_2y_3 \) from:

\[(y_1 + y_2 + y_3)^2 - (y_1^2 + y_2^2 + y_3^2) = 2(y_1y_2 + y_1y_3 + y_2y_3).\]

Get \( y_1y_2y_3 \) since its equal to:

\[
\frac{(y_1y_2+y_1y_3+y_2y_3)(y_1+y_2+y_3)-(y_1+y_2+y_3)(y_1^2+y_2^2+y_3^2)+(y_1^3+y_2^3+y_3^3)}{3}.
\]
Solution Two

Bob computes (next slide shows how)

\[
\sum_{1 \leq i \leq n-3} x_i
\]

\[
\sum_{1 \leq i < j \leq n-3} x_i x_j
\]

\[
\sum_{1 \leq i < j < k \leq n-3} x_i x_j x_k
\]

From these CAN get (next next slide shows how)

\[
y_1 + y_2 + y_3, \quad y_1 y_2 + y_1 y_3 + y_2 y_3, \quad y_1 y_2 y_3
\]

Cleanly!
Bob Can Actually Compute Those Sums

Let

\[ s^L_0(x_1, \ldots, x_L) = 1 \] (For Notational Niceness.)
\[ s^L_1(x_1, \ldots, x_L) = \sum_{1 \leq i \leq L} x_i \]
\[ s^L_2(x_1, \ldots, x_L) = \sum_{1 \leq i < j \leq L} x_i x_j \]
\[ s^L_3(x_1, \ldots, x_L) = \sum_{1 \leq i < j < k \leq L} x_i x_j x_k \]

Let \( s^L_i \) mean \( s^L_i(x_1, \ldots, x_L) \).

We show that if Bob has

\[ s^{L-1}_0, \quad s^{L-1}_1, \quad s^{L-1}_2, \quad s^{L-1}_3, \quad x_L. \]

then he can compute

\[ s^L_0, \quad s^L_1, \quad s^L_2, \quad s^L_3. \]

\[ s^L_0 = 1 \]
\[ s^L_1 = s^{L-1}_1 + x_L s^{L-1}_0 \]
\[ s^L_2 = s^{L-1}_2 + x_L s^{L-1}_1 \]
\[ s^L_3 = s^{L-1}_3 + x_L s^{L-1}_2 \]
Getting $s_i^3(y_1, y_2, y_3)$ from $s_i^{n-3}$

Let $s_i^3$ mean $s_i^3(y_1, y_2, y_3)$. One can show:

$$s_1^n = s_1^{n-3} s_0^3 + s_0^{n-3} s_1^3$$
$$s_2^n = s_2^{n-3} s_0^3 + s_1^{n-3} s_1^3 + s_0^{n-3} s_2^3$$
$$s_3^n = s_3^{n-3} s_0^3 + s_2^{n-3} s_1^3 + s_1^{n-3} s_2^3 + s_3^{n-3} s_0^3$$

Bob knows

$$s_1^n, \quad s_1^{n-3}, \quad s_2^n, \quad s_2^{n-3}, \quad s_3^n, \quad s_3^{n-3}$$

By solving three linear equations in three variables he can find:

$$s_1^3, \quad s_2^3, \quad s_3^3$$