Open Problems Column
Edited by William Gasarch

This Issue's Column! This issue's Open Problem Column is by William Gasarch, Emily Kaplitz, and Erik Metz. It is on the mod behaviour of the sequence

$a_1 = 1$

$(\forall n \geq 2)[a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}]$.

Request for Columns! I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be (1) broad or narrow or anywhere inbetween, and (2) really important or really unimportant or anywhere inbetween.

How Does $a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$ Behave Mod M?
By William Gasarch[1] and Emily Kaplitz[2] and Erik Metz[3]

1 The Sequence

In the book *Sequences and Mathematical Induction in Mathematical Olympiad Competitions* [1], on page 7, is the following problem:

> The sequence a_n is defined by
> $a_1 = 1$
> $(\forall n \geq 2)[a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}]$.
> Prove that there are infinitely many terms of the sequence that are divided by 7.

We will henceforth refer to the sequence a_n defined above as *the sequence*.

We give their proof and then make some observations and conjectures.

Theorem 1.1 $(\forall m \geq 1)(\exists i_1 < \cdots < i_m)[a_{i_1} \equiv \cdots \equiv a_{i_m} \equiv 0 \pmod{7}]$.

Proof:

Throughout this proof $\equiv$ means $\equiv \pmod{7}$.

The proof is by induction on m.

Base Case $m = 1$: Note that $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$, $a_5 = 7$. Hence $i_1 = 5$ suffices.

Induction Hypothesis: $(\exists i_1 < \cdots < i_{m-1})[a_{i_1} \equiv \cdots \equiv a_{i_{m-1}} \equiv 0 \pmod{7}]$.

Induction Step: Let $n = i_{m-1}$. Note that

$a_{2n} = a_{2n-1} + a_n \equiv a_{2n-1}$.

$a_{2n+1} = a_{2n} + a_n \equiv a_{2n}$.

Combining these we obtain that there is an $r \in \{0, \ldots, 6\}$ such that

$$a_{2n-1} \equiv a_{2n} \equiv a_{2n+1} \equiv r.$$

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[1] gasarch@umd.edu
[2] ekaplitz@umd.edu
[3] emetz1618@gmail.com

Case 1: $r = 0$. Then $a_{2n-1} \equiv 0$ so we can take $i_m = 2n - 1$.

Case 2: $r \neq 0$. Look at the following numbers
$a_{4n-3} \equiv a_{4n-3} + 0$ (You will see later why we include this.)
$a_{4n-2} = a_{4n-3} + a_{2n-1} \equiv a_{4n-3} + r$.
$a_{4n-1} = a_{4n-2} + a_{2n-1} \equiv a_{4n-3} + 2r$.
$a_{4n} = a_{4n-1} + a_{2n} \equiv a_{4n-3} + 3r$.
$a_{4n+1} = a_{4n} + a_{2n} \equiv a_{4n-3} + 4r$.
$a_{4n+2} = a_{4n+1} + a_{2n+1} \equiv a_{4n-3} + 5r$.
$a_{4n+3} = a_{4n+2} + a_{2n+1} \equiv a_{4n-3} + 6r$.
Since $r \not\equiv 0$, and 7 is prime, the numbers $0, r, 2r, 3r, 4r, 5r, 6r$ are equivalent to all 7 possibilities mod 7. Hence for some $i \in \{4n - 3, \ldots, 4n + 3\}$, $a_i \equiv 0$. We set i_m to i. ∎

2   Other Mods and Empirical Evidence

We leave it to the reader to adapt the proof of Theorem 1.1 to show the following:

Theorem 2.1 *Let $r \in \{2, 3, 5, 7\}$. $(\forall m \geq 1)(\exists i_1 < \cdots < i_m)[a_{i_1} \equiv \cdots \equiv a_{i_m} \equiv 0 \pmod{r}]$.*

The question arises:
Find all r such that

$$(\forall m \geq 1)(\exists i_1 < \cdots < i_m)[a_{i_1} \equiv \cdots \equiv a_{i_m} \equiv 0 \pmod{r}].$$

We wrote a program to generate the first million elements of the sequence $a_n \pmod{r}$ for all $r = 2$ to 100. The empirical evidence strongly suggests the following conjecture:

Conjecture 2.2

1. *Let $r \not\equiv 0 \pmod{4}$. Then*

$$(\forall m \geq 1)(\exists i_1 < \cdots < i_m)[a_{i_1} \equiv \cdots \equiv a_{i_m} \equiv 0 \pmod{r}].$$

2. *Let $r \equiv 0 \pmod{4}$. Then*

$$(\forall m \geq 1)(a_m \not\equiv 0 \pmod{r}).$$

The second part should not be called a conjecture since we prove it in the next section.

3 $r \equiv 0 \pmod 4 \implies (\forall m \geq 1)[a_m \not\equiv 0 \pmod r]$

In this section we will prove that, for all m, $a_m \not\equiv 0 \pmod 4$. A trivial corollary is

$$r \equiv 0 \pmod 4 \implies (\forall m \geq 1)[a_m \not\equiv 0 \pmod r].$$

The first few terms of the sequence mod 4 are

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
1	2	3	1	3	2	1	2	3	2	1	3

This pattern indicates three things:

- If n is odd then $a_n \equiv 1, 3 \pmod 4$.

- If you remove the 2's from the sequence you get $1, 3, 1, 3, 1, 3$

- $a_n \not\equiv 0 \pmod 4$.

We prove all three.

Theorem 3.1 *All $\equiv$ are mod 4.*

1. *$(\forall n \geq 1)[$If n is odd then $a_n \equiv 1, 3 \pmod 4]$*

2. *$a_1 \not\equiv 0$ and $a_2 \not\equiv 0$. (We separate these two cases since Part 2 only covers $n \geq 3$.)*

3. *$(\forall n \geq 3)$*

   (a) If $a_n \equiv 1$ then either $a_{n-1} \equiv 3$ or $a_{n-1} \equiv 2$ and $a_{n-2} \equiv 3$.
   (b) If $a_n \equiv 3$ then either $a_{n-1} \equiv 1$ or $a_{n-1} \equiv 2$ and $a_{n-2} \equiv 1$.
   (c) $a_n \not\equiv 0$.

Proof: The following equations will be used throughout and are easily verified.
   EQ1: $a_{2m-1} = a_{2m-2} + a_{m-1}$
   EQ2: $a_{2m} = a_{2m-1} + a_m$
   EQ3: $a_{2m+1} = a_{2m} + a_m$
   EQ4: $a_{2m+1} = a_{2m-1} + 2a_m$.

1) We prove this by induction by a triple induction on n.
Base Case $n = 1$. $a_1 = 1 \equiv 1$.

IH For all $1 \leq n' < n$, if n' is odd then $a_{n'} \equiv 1, 3$.

IS If n is even there is nothing to prove, so we take n to be odd. Let $n = 2m + 1$. By the IH, $a_{2m-1} \equiv 1, 3$. By EQ4 $a_{2m+1} = a_{2m-1} + 2a_m$, hence since $a_{2m-1} \equiv 1, 3$ we have $a_{2m+1} \equiv 1, 3$.

2) This is obvious.

3) We prove this by induction on n. We will assume the theorem for all $3 \leq n' \leq n - 1$ and n even, and prove it for n and $n + 1$.

Base Case The theorem starts at $n = 3$. From the table of a_i's before this theorem one can see that the theorem holds for $n = 3, 4, 5, 6, 7$. So the proof below needs to work for $n \geq 8$. You will see at the proof of 3a why we needed to start at $n = 8$.

IH For all n', $3 \leq n' \leq n - 1$, all three parts of the theorem holds. Note that $3 < n - 1$ since $n \geq 8$.

IS Let $n = 2m$ and $n \geq 8$. Note that $m \geq 4$. We prove 3a for a_{2m}. The proof for 3b is similar.

3a) By Part 1, $a_{2m-1} \not\equiv 0$, hence $a_{2m-1} \equiv 1, 3$. We will do the $a_{2m-1} \equiv 1$ case and leave the $a_{2m-1} \equiv 3$ case to the reader. We have cases based on a_m. By the IH (Part 3c), $a_m \equiv 1, 2, 3$. (Need that $m \geq 1$ to use IH, and we have $m \geq 4$.)

    Case 1 $a_m \equiv 1$.

$$\text{EQ2: } a_{2m} = a_{2m-1} + a_m \equiv 1 + 1 \equiv 2.$$

$$\text{EQ3: } a_{2m+1} = a_{2m} + a_m \equiv 2 + 1 \equiv 3.$$

Case 2 $a_m \equiv 2$.

$$\text{EQ2: } a_{2m} = a_{2m-1} + a_m \equiv 1 + 2 \equiv 3.$$

$$\text{EQ3: } a_{2m+1} = a_{2m} + a_m \equiv 3 + 2 \equiv 1.$$

    Case 3 $a_m \equiv 3$. We show this case cannot occur. Since $a_m \equiv 3$, by the IH, either (1) $a_{m-1} \equiv 1$ or (2) $a_{m-1} \equiv 2$ and $a_{m-2} \equiv 1$. (Need $m \geq 3$ to use the IH, and we have $m \geq 4$.)

    Case 3.1 $a_{m-1} \equiv 1$.

$$\begin{aligned} \text{EQ1:} \quad a_{2m-1} &= a_{2m-2} + a_{m-1} \\ 1 &\equiv a_{2m-2} + 1 \\ a_{2m-2} &\equiv 0 \text{ which contradicts the IH.} \end{aligned}$$

(Need $2m - 2 \geq 3$ to use the IH, and we have $m \geq 4$.)

    Case 3.2 $a_{m-1} \equiv 2$ and $a_{m-2} \equiv 1$.

We recap and extend what we know.

We are assuming $a_{2m-1} \equiv 1$.

From EQ1,

$$a_{2m-1} = a_{2m-2} + a_{m-1}.$$

Putting in $a_{2m-1} \equiv 1$ and $a_{m-1} \equiv 2$, we get

$$a_{2m-2} \equiv 3.$$

From the recurrence, we have

$$a_{2m-2} = a_{2m-3} + a_{m-1}.$$

Putting in $a_{2m-2} \equiv 3$ and $a_{m-1} \equiv 2$, we have

$$3 \equiv a_{2m-3} + 2,$$

so

$$a_{2m-3} \equiv 1.$$

From the recurrence, we have

$$a_{2m-3} = a_{2m-4} + a_{m-2}.$$

Putting in $a_{2m-3} \equiv 1$ and $a_{m-2} \equiv 1$, we get

$$1 = a_{2m-4} + 1,$$

so

$$a_{2m-4} \equiv 0.$$

This contradicts the IH. (Need $2m - 4 \geq 3$ to use the IH, and we have $m \geq 4$. Note that $m \geq 3$ would not have sufficed.)

3b) The proof is similar to that of Part 3a.

3c) We prove this in the reverse order: we first show $a_{n+1} \not\equiv 0$ and then that $a_n \not\equiv 0$.

a_{n+1}: Since n is even, $n + 1$ is odd. By Part 1 $a_{n+1} \equiv 1, 3$, hence $a_{n+1} \not\equiv 0$.

a_n: Since $a_{n+1} \equiv 1, 3$, by Part 3b (not the IH, but what I just proved in the IS), $a_n \equiv 1, 2, 3$, so $a_n \not\equiv 0$. ∎

4 Conclusion

We restate the part of our conjecture that is still unproven:

Conjecture 4.1 *Let $r \not\equiv 0 \pmod 4$. Then*

$$(\forall m \geq 1)(\exists i_1 < \cdots < i_m)[a_{i_1} \equiv \cdots \equiv a_{i_m} \equiv 0 \pmod r].$$

The following questions can also be considered

1. Let $r \not\equiv 0 \pmod 4$. Let $0 \leq i \leq r - 1$. What is the density of

$$\{a_n : a_n \equiv i \pmod r\}.$$

2. Let $A, B, C \in \mathbb{Z}$ and $r \geq 2$. What is the behaviour of

    $a_1 = A.$

    $a_n = Ba_{n-1} + Ca_{\lfloor n/2 \rfloor} \pmod r.$

References

[1] Z. Feng, editor. *Sequences and Mathematical Induction In Mathematical Olympiad Competitions.* World Scientific, Singapore, 2020. Translated by Feng Ma and Youren Wang.