Open Problems Column
Edited by William Gasarch

I am delighted to be the new Open Problems Column Editor. I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be (1) broad or narrow or anywhere inbetween, and (2) really important or really unimportant or anywhere inbetween.

Open Problems About
Grid Coloring and The Complexity of Grid Colorings
By William Gasarch

1 The Grid Coloring Problem

Notation 1.1 If $n \in \mathsf{N}$ then $[n] = \{1, \ldots, n\}$. If $n, m \in \mathsf{N}$ then $G_{n,m}$ is the grid $[n] \times [m]$.

Def 1.2 A *rectangle* of $G_{n,m}$ is a subset of the form $\{(a, b), (a + c_1, b), (a + c_1, b + c_2), (a, b + c_2)\}$ for some $a, b, c_1, c_2 \in \mathsf{N}$. A grid $G_{n,m}$ is *c-colorable* if there is a function $\chi : G_{n,m} \to [c]$ such that there are no rectangles with all four corners the same color.

Not all grids have c-colorings. As an example, for any c clearly $G_{c+1,c^{c+1}+1}$ does not have a c-coloring by two applications of the pigeonhole principle. The following question has been studied [3, 4]. We follow the approach and notation in the paper by Fenner et al [4].

The main question is:

For which values of n and m is $G_{n,m}$ c-colorable?

Def 1.3 Let $n, m, n', m' \in \mathsf{N}$. $G_{m,n}$ *contains* $G_{n',m'}$ if $n' \le n$ and $m' \le m$. $G_{m,n}$ *is contained in* $G_{n',m'}$ if $n \le n'$ and $m \le m'$. Proper containment means that at least one of the inequalities is strict.

Clearly, if $G_{n,m}$ is c-colorable, then all grids that it contains are c-colorable. Likewise, if $G_{n,m}$ is not c-colorable then all grids that contain it are not c-colorable.

Def 1.4 Fix $c \in \mathsf{N}$. OBS_c is the set of all grids $G_{n,m}$ such that $G_{n,m}$ is not c-colorable but all grids properly contained in $G_{m,n}$ are c-colorable. OBS_c stands for *Obstruction Sets*.

We leave the proof of the following theorem to the reader.

Theorem 1.5 *Fix $c \in \mathsf{N}$. A grid $G_{n,m}$ is c-colorable iff it does not contain any element of* OBS_c.

By Theorem 1.5 we can rephrase the question of finding which grids are c-colorable:

What is OBS_c*?*

Note that if $G_{n,m} \in \mathrm{OBS}_c$, then $G_{m,n} \in \mathrm{OBS}_c$.

The problem of which grids can be colored is clearly within Ramsey Theory. In the last section I will comment on other problems in Ramsey Theory that our questions can be asked of.

2 Some Results about Grid Coloring

Fenner et al [4] has many general theorems that allow one to find some of the sets OBS_n.

The following theorem was proven just from the tools:

Theorem 2.1 $\mathrm{OBS}_2 = \{G_{7,3}, G_{5,5}, G_{3,7}\}$.

The following theorem was proven mostly from the tools; however, (1) 3-coloring of $G_{10,10}$ was found with a computer program, and (2) the proof that $G_{10,11}$ is not 3-colorable was a case analysis.

Theorem 2.2

$$\mathrm{OBS}_3 = \{G_{19,4}, G_{16,5}, G_{13,7}, G_{11,10}, G_{10,11}, G_{7,13}, G_{5,16}, G_{4,19}\}.$$

The following theorem was proven partially from the tools; however, (1) 4-colorings of $G_{24,9}$, $G_{22,10}$, $G_{21,12}$, $G_{17,17}$, $G_{18,18}$ were found by a computer program (see [4] for who did what), and (2) the proof that $G_{19,17}$ is not 4-colorable was a case analysis.

Theorem 2.3

$$\mathrm{OBS}_4 = \{G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,10}, G_{22,11}, G_{21,13}, G_{19,17}\} \bigcup$$

$$\{G_{17,19}, G_{13,21}, G_{11,22}, G_{10,23}, G_{9,25}, G_{7,29}, G_{6,31}, G_{5,41}\}$$

The 4-coloring of $G_{17,17}$ was particularly difficult to find. So much so that on November 30, 2009 I posted the following challenge on Complexity Blog [5] which I paraphrase.

BEGIN PARAPHRASE
The 17×17 **challenge:** The first person to email me a 4-coloring of $G_{17,17}$ in LaTeX will win \$289.00. (289.00 is chosen since it is 17^2.)
END PARAPHRASE

Brian Hayes, the Mathematics columnist for Scientific American, publicized the challenge [10]. Initially there was a lot of activity on the problem. Some used SAT solvers, some used linear programming, and one person offered an exchange: *buy me a \$5000 computer and I'll solve it.* Finally in 2012 Bernd Steinbach and Christian Posthoff [11, 7] solved the problem. They used a rather clever algorithm with a SAT solver. They believed that the solution was close to the limits of their techniques. They also gave me a coloring of $G_{18,18}$ gratis.

Though this particular instance of the problem was solved, the problem of grid coloring in general seems to be difficult. We come back to this point in Section 4.

3 Open Problems on Grid Coloring

1. Refine the tools so that the case by case analysis and the grid colorings found by computer search can be be corollaries

2. We know that $2\sqrt{c}(1 - o(1)) \le |\mathrm{OBS}_c| \le 2c^2$. Bring these bounds closer together.

3. All of the results of the form $G_{n,m}$ *is not c-colorable* have the same type of proof: show that there is no rectangle free subset of $G_{n,m}$ of size $\lceil ab/c \rceil$. Either

   - show that if a grid $G_{n,m}$ has a rectangle free set of size $\lceil nm/c \rceil$ then it is c-colorable, or
   - develop some other technique to show grids are not c-colorable.

4. Find OBS_5 and beyond!

4 The Complexity of Grid Colorings

Why was finding a 4-coloring of $G_{18,18}$ so hard? Can we formalize this question?

Consider this set:

$$A = \{(1^n, 1^m, 1^c) : G_{n,m} \text{ is } c\text{-colorable } \}.$$

Clearly $A \in$ NP; however, it is unlikely to be NP-complete since A is sparse (its known that if a sparse set is NP-complete then P = NP). We will now do a standard work-around: we define a different set. We also caution that we are straying from our original problem which is one of the points of this column.

I once again offered a challenge through complexity blog but this time with no cash award. We paraphrase the post [6].

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Def 4.1 Let $c, n, m \in \mathsf{N}$.

1. A mapping χ of $G_{N,M}$ to $[c]$ is a *c-coloring* if there are no monochromatic rectangles.
2. A partial mapping χ of $G_{n,m}$ to $[c]$ is *extendable to a c-coloring* if there is an extension of χ to a total mapping which is a c-coloring of $G_{N,M}$. We will use the term *extendable* if the c is understood.

Def 4.2 Let

$$GCE = \{(n, m, c, \chi) \mid \chi \text{ is extendable}\}.$$

GCE stands for *Grid Coloring Extension.*

CHALLENGE: Prove that GCE is NP-complete and fame and fortune will be yours! Well, maybe not fortune. Well, maybe not fame either.

END PARAPHRASE

Brian Lawler proved it was NP-complete, Daniel Apon and myself added to the result, and we have a paper [1].

5 Open Problems in the Complexity of Grid Colorings

The result *GCE is* NP*-complete* seems to have wandered off from our original goal. Determining if a a given coloring can be extended is hard; however, it may be possible that in the once case we care about, starting with the empty coloring, the problem is easy.

The first open problem is to determine the complexity of

$$A = \{(1^n, 1^m, 1^c) : G_{n,m} \text{ is } c\text{-colorable}\}.$$

We suspect that A is hard but do not even know how to state it. A is in NP but is likely not NP-complete since its a sparse set.

The second open problem is to determine the complexity of

$$B = \{(1^n, 1^c) : G_{n,n} \text{ is } c\text{-colorable}\}.$$

For this one we have no opinion- it may actually be easy.

Speculation on why B might be in P*.*

It is possible that some nice mathematics will yield a characterization of when $G_{n,n}$ is c-colorable. In fact the following are known [4].

- If c is a prime power then G_{c^2,c^2+c} is c-colorable.
- For all c, G_{c^2,c^2+c+1} is not c-colorable.
- Hence, for c a prime power, (a) if $x \le c^2$ then $(1^x, 1^c) \in B$, but (b) if $x \ge c^2 + c + 1$ then $(1^x, 1^c) \notin B$. Hence there are not that many hard cases. One can imagine the gap being closed and extended to all c.

Speculation on why B might not be in P*.*

Recall that what started us thinking about the complexity of grid coloring was that showing $G_{17,17}$ is 4-colorable was so hard. We currently have some high school students looking at 5-coloring and again the square case seems hard.

The third open problem is more of an open ended question. Recall earlier when I said *We suspect that A is hard but do not even know how to state it.* This is really the issue. We need a complexity theory of sparse sets aimed at problems like A and B above. And if we do, will the problems above be shown hard?

6 Other Problems in Ramsey Theory

The following is known:

For all c there is a number $n = n(c)$ such that, for all c-colorings of $G_{n,n}$ there is a monochromatic square.

This theorem leads to open problems similar to the ones above. There is one difference.

- For colorings that avoid mono rectangles we know reasonable bounds. For example any c-coloring of G_{c^2+c+1,c^2} has a mono rectangle. Hence the function, given c, return the least n such that $G_{n,n}$ is not c-colorable (avoiding mono rectangles), has output around $O(c^2)$. If its hard to compute it won't be because the output is too long.

- For colorings that avoid mono squares the best known upper bounds are enormous (the bound $2^{2^{2^{2^c}}}$ can be derived from [9]) but the reality might be much smaller. It is known that $G(2) = 15$ [2] but nothing else is known. None of the obstruction sets are known. The function, given c, return the least n such that $G_{n,n}$ is not c-colorable (avoiding mono squares), might have an enormous output. If its hard to compute it may be because the output is too long. Hence we would need to use a complexity theory based on the size of the input and the output.

More generally, many problems in Ramsey theory use bounds that are, to quote Graham, Rothchild, and Spencer [8] EEEEEEEEEEEEENORMOUS!!!! Our grid coloring problem avoids that issue. The Ramsey Numbers themselves are not quite so bad- for c colors they are roughly $c^{\log c}$.

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