Open Problems Column
Edited by William Gasarch
This Issue's Column!

This issue's Open Problem Column is by William Gasarch, Nathan Hayes, Anthony Ostuni, and Davin Park. It is *The complexity of chromatic number when restricted to graphs with either bounded genus or bounded crossing number.*

Request for Columns!

I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be:

1. Broad, narrow, or anywhere in between.

2. Important, unimportant, or anywhere in between.

The Complexity of Chromatic Number When Restricted to graphs with Bounded Genus or Bounded Crossing Number

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1 Introduction

Convention 1.1 Throughout this paper if $G = (V, E)$ is a graph then $n = |V|$ and $m = |E|$.

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Def 1.2 Let $G = (V, E)$ be a graph. The *chromatic number of* G is the least c such that V can be c-colored in such a way that no two adjacent vertices have the same color. We denote this by $\chi(G)$.

Def 1.3 Let $\mathcal{G}$ be a class of graphs and $r \in \mathsf{N}$. The problem

$$\{G : G \in \mathcal{G} \wedge \chi(G) \leq r\}$$

is *trivial* if $\mathcal{G} \in \mathrm{P}$ and all graphs in $\mathcal{G}$ have $\chi(G) \leq r$.

We present a known theorem with the following authors. Part 1 was proven by Hopcroft and Tarjan [10]. Part 2 is easy. Part 3 was proven by Garey, Johnson, and Stockmeyer [6]. Part 4 was proven by Appel, Haken, and Koch [2, 3].

Theorem 1.4

1. *The following problem is in* $O(n)$ *time:* $\{G : G \text{ is planar}\}$.

   Hence if

   $$\{G : G \text{ is planar and } \chi(G) \leq r\}$$

   is NP*-complete, the difficulty lies in the coloring, not the planarity.*

2. $\{G : G \text{ is planar and } \chi(G) \leq 2\}$ *is in* P *(also true without the planar restriction).*

3. $\{G : G \text{ is planar and } \chi(G) \leq 3\}$ *is* NP*-complete.*

4. *All planar graphs are 4-colorable. Hence* $\{G : G \text{ is planar and } \chi(G) \leq 4\}$ *is trivial.*

What about other restrictions on graphs?

Def 1.5 Let $G = (V, E)$ be a graph. The *genus of* G is the least g such that G can be drawn on a sphere with g handles with no edges crossing. Note that a planar graph has genus 0. We denote the genus of G by $\mathrm{g}(G)$.

Def 1.6 Let $G = (V, E)$ be a graph. The *crossing number of* G is the least c such that G can be drawn in the plane with c edges crossing. Note that a planar graph has crossing number 0. We denote the crossing number of G by $\mathrm{cr}(G)$.

In this paper we raise the following questions:

1. For which (g, r) is the following problem in P? NP-complete? Trivial?

$$\{G : \mathrm{g}(G) \le g \wedge \chi(G) \le r\}.$$

2. For which (c, r) is the following problem in P? NP-complete? Trivial?

$$\{G : \mathrm{cr}(G) \le c \wedge \chi(G) \le r\}.$$

We will first prove a very general theorem that will allow us to prove many sets of graphs are in P. We will then consider the cases of bounded genus and bounded crossing number separately and state which problems are in P, NP-complete, trivial, and open.

2 Graphs with $m \le \frac{10n}{3} + O(1)$

Def 2.1 A graph H is *r-critical* if $\chi(H) = r$ but for all subgraphs H' of H, $\chi(H') \le r - 1$.

Kostochka and Yancey [12] proved the following.

Theorem 2.2 *Let $H = (V, E)$. Let $r \ge 4$.*

1. *If H is r-critical then*

$$m \ge \left\lceil \frac{(r+1)(r-2)n - r(r-3)}{2(r-1)} \right\rceil.$$

2. *Let $r \ge 7$. If H is r-critical then $m \ge \frac{10n}{3} - \frac{5}{3}$. (Follows from Part 1.)*

Def 2.3 A class of graphs $\mathcal{G}$ is *awesome* if the following three hold:

1. $\mathcal{G}$ is in P.
2. $\mathcal{G}$ is closed under subgraphs.
3. There exists $\alpha < \frac{10}{3}$ and β such that, for all $G \in \mathcal{G}$, $m \le \alpha n + \beta$.

Lemma 2.4 *Let $c \geq 0$, and $r \geq 6$. Let $\mathcal{G}$ be an awesome class of graphs with parameters α, β. Let $A = \frac{(7/3)+\beta}{(10/3)-\alpha}$.*

1. *For all $(r+1)$-critical $H \in \mathcal{G}$, $n \leq A$.*

2. *If $\chi(G) \geq r+1$ then there is a subgraph H of G on $\leq A$ vertices such that $\chi(H) \geq r+1$.*

3. *If for all subgraphs H of G on $\leq A$ vertices, $\chi(H) \leq r$, then $\chi(G) \leq r$. This is just the contrapositive of Part 2.*

Proof:
1) Since $H \in \mathcal{G}$, $m \leq \alpha n + \beta$. By Theorem 2.2.2, $m \geq \frac{10n}{3} - \frac{7}{3}$. Hence

$$\frac{10n}{3} - \frac{7}{3} \leq m \leq \alpha n + \beta$$

By algebra we get $n \leq A$.

2) Let H be a $(r+1)$-critical subgraph of G. By Part 1, H has $\leq A$ vertices. Since H is $(r+1)$-critical $\chi(H) = r+1$. ∎

Theorem 2.5 *Let $\mathcal{G}$ be an awesome set of graphs and let $r \geq 6$. The following problem is in* P*:*

$$\{G : G \in \mathcal{G} \wedge \chi(G) \leq r\}.$$

Proof:

Let $\mathcal{G}$ be awesome with parameters α, β. Let $A = \frac{(7/3)+\beta}{(10/3)-\alpha}$.

1. Input G.

2. Test if $G \in \mathcal{G}$ (this can be done in polynomial time since $\mathcal{G}$ is awesome). If $G \notin \mathcal{G}$ then output NO and halt.

3. For all subgraphs H of G on $\leq A$ vertices determine if $\chi(H) \leq r$ by brute force. Note that there are $\sum_{i=0}^{A} \binom{n}{i} \leq n^{A+1}$ such subgraphs to check and each check takes $\leq r^A$ steps to check, so the total time used is $\leq r^A n^{A+1}$, a polynomial.

4. If there is a subgraph H with $\chi(H) \geq r+1$ then $\chi(G) \geq r+1$ so the answer is NO. If there is no such subgraph H then, by Lemma 2.4.3, $\chi(G) \leq r$ so the answer is YES.

∎

3 Graphs with Bounded Genus

3.1 If $r \geq 6$ then $\chi(G) \leq r$ is in P

Mohar [13, 14] proved the following.

Theorem 3.1 *Fix g. The following problem is in $O(n)$ time (the constant will depend on g): $\{G : \mathrm{g}(G) \leq g\}$.*

Hence if

$$\{G : \mathrm{g}(G) \leq g \wedge \chi(G) \leq r\}$$

is NP*-complete, the difficulty will lie in the coloring, not the bound on* $\mathrm{g}(G)$*.*

The following theorem is well known. We include it and its proof for completeness.

Theorem 3.2 *Let G be a graph. If $\mathrm{g}(G) \leq g$ then $m \leq 3n + 6g - 6$.*

Proof: Let G be a graph of genus g. Draw it on a surface of genus g. Let f be the number of faces. Euler showed that

$$n - m + f = 2 - 2g.$$

Let $F_1, \ldots, F_f$ be the faces and let m_i be the number of edges bounding face F_i. Note that $m_i \geq 3$ and $\sum_{i=1}^{f} m_i$ counts every edge twice. Hence

$$2m = \sum_{i=1}^{f} m_i \geq \sum_{i=1}^{f} 3 = 3f$$

so $f \leq \frac{2m}{3}$. Hence

$$2 - 2g = n - m + f \leq n - m + \frac{2m}{3} = n - \frac{m}{3}.$$

By algebra we get $m \leq 3n + 6g - 6$.

$\blacksquare$

Theorem 3.3 *Let $g \geq 0$ and $r \geq 5$. The following problem is in* P*:*

$$\{G : \mathrm{g}(G) \leq g \wedge \chi(G) \leq r\}.$$

Proof: We consider the $r \geq 6$ case and the $r = 5$ case separately.

$\boldsymbol{r \geq 6}$:

We first show that $\mathcal{G} = \{G : \mathrm{g}(G) \leq g\}$ is an awesome set (see Definition 2.3). We go through all three properties needed.

- By Theorem 3.1, $\mathcal{G} \in \mathrm{P}$.
- Clearly if $\mathrm{g}(G) \leq g$ and H is a subgraph of G, then $\mathrm{g}(H) \leq g$.
- Let $\alpha = 3$ and $\beta = 6g - 6$. By Theorem 3.2 $m \leq 3n + (6g - 6)$ where m is the number of edges in G. Hence G satisfies the third condition of being awesome with $\alpha = 3$ and $\beta = 6g - 6$.

By Theorem 2.5, if $\mathcal{G}$ is an awesome set of graphs and $r \geq 6$ then

$$\{G : G \in \mathcal{G} \wedge \chi(G) \leq r\} \in \mathrm{P}.$$

Hence

$$\{G : \mathrm{g}(G) \leq g \wedge \chi(G) \leq r\} \in \mathrm{P}.$$

$\boldsymbol{r = 5}$:

Thomassen [23] (see also the book by Mohar and Thomassen [16, Corollary 8.4.9]) proved that

$$\{G : \mathrm{g}(G) \leq g \wedge \chi(G) \leq 5\} \in \mathrm{P}.$$

The proof is rather difficult. They do not give a time bound.

∎

3.2 When is the Problem Trivial?

For the next theorem: Heawood [9] proved the $g \geq 1$ case of Part 1; however, Appel-Haken-Koch [2, 3] proved the $g = 0$ case (this is the 4-color theorem). Ringel and Young [19] proved Part 2.

Theorem 3.4

1. *If G is a graph of genus g then*

$$\chi(G) \leq \left\lfloor \frac{7+\sqrt{49-24(2-2g)}}{2} \right\rfloor.$$

(The quantity $2-2g$ is the Euler Characteristic*, denoted $e(G)$, hence this theorem is often stated with $e(G)$ instead of $2-2g$.)*

2. *For $g \geq 0$ there exists a graph of genus g such that*

$$\chi(G) \geq \left\lfloor \frac{7+\sqrt{49-24(2-2g)}}{2} \right\rfloor.$$

Hence the bound in Part 1 is tight.

3.3 Summary of What is Known for Bounded Genus

We summarize the complexity of the set

$$\mathrm{COL}_{g,r} = \{G : \mathrm{g}(G) \leq g \wedge \chi(G) \leq r\}.$$

1. If $r = 2$ and $g \geq 0$ then $\mathrm{COL}_{g,r}$ is in P. (Theorem 1.4.2)
2. If $r = 3$ and $g \geq 0$ then $\mathrm{COL}_{g,r}$ is NP-complete. (Theorem 1.4.3)
3. If $r = 4$ and $g = 0$ then $\mathrm{COL}_{g,r}$ is trivial. (Theorem 1.4.4)
4. If $r = 5$ and $g \geq 0$ then $\mathrm{COL}_{g,r}$ is in P. (Theorem 3.3)
5. If $r \geq 6$ then $\mathrm{COL}_{g,r} \in$ P, but see next point. (Theorem 3.3)
6. For $g \geq 0$, for $r \geq \left\lfloor \frac{7+\sqrt{49-24(2-2g)}}{2} \right\rfloor$, $\mathrm{COL}_{g,r}$ is trivial. (Theorem 3.4)
7. For $g \geq 0$, for $r = \left\lfloor \frac{7+\sqrt{49-24(2-2g)}}{2} \right\rfloor - 1$, $\mathrm{COL}_{g,r}$ is not trivial. (Theorem 3.4)

Erman et al. [5, Problem 6.4] asks about genus $g \geq 1$ and 4-coloring. Here is a quote from that paper:

For any fixed surface, does there exist a polynomial time algorithm for deciding if a graph embeddable on this surface is 4-colorable?

Mohar and Thomassen [16, Problem 8.4.10] also ask about genus $g \geq 1$ and 4-coloring. Here is a quote from that book:

Let S be a fixed surface. Does there exist a polynomially bounded algorithm for deciding if a graph on surface S can be 4-colored?

4 Graphs with Bounded Crossing Number

For more information about crossing numbers and proofs of some of the theorems we state, see the book by Schaefer [22] and/or the survey by Schaefer [21].

Kawarabayashi and Reed [11] proved the following.

Theorem 4.1 *Fix c. The following problem is in $O(n)$ time (the constant will depend on c):*

$$\{G : \mathrm{cr}(G) \leq c\}.$$

Theorem 4.1 is important for locating why

$$\{G : \mathrm{cr}(G) \leq c \wedge \chi(G) \leq r\}$$

is hard, if it is hard. More precisely, if that set is NP-complete, the difficulty will lie in the coloring, not the bound on $\mathrm{cr}(G)$.

We will be able to use some of the result about genus in our study of crossing numbers by using the following known (and easy) lemma.

Lemma 4.2 *For all graphs G, $\mathrm{g}(G) \leq \mathrm{cr}(G)$.*

4.1 If $r \geq 5$ then $\chi(G) \leq r$ is in P

Theorem 4.3 *Fix c. Fix $r \geq 5$. The set*

$$\{G : \mathrm{cr}(G) \leq c \wedge \chi(G) \leq r\}$$

is in P*.*

Proof: The following is a polynomial time algorithm for the problem.

1. Input G.
2. Test if $\mathrm{cr}(G) \le c$ (this can be done in linear time by Theorem 4.1). If NO then output NO and halt.
3. (If the algorithm got here then $\mathrm{cr}(G) \le c$. By Lemma 4.2, $\mathrm{g}(G) \le c$.) Run the algorithm from Theorem 3.3 to determine if $\chi(G) \le r$.

∎

4.2 If $r \ge 6$ then $\chi(G) \le r$ is in P

We give an alternative proof of Theorem 4.3 in the $r \ge 6$ case. The following theorem is well known. We include it and its proof for completeness.

Theorem 4.4 *If G is a graph with $\mathrm{cr}(G) \le c$ then $m \le 3n + c$.*

Proof: Take graph G and, for each crossing, remove one of the edges. The resulting graph is planar, so the number of edges is bounded by thrice the number of vertices. Hence $m - c \le 3n$, so $m \le 3n + c$. ∎

Theorem 4.5 *Let $c \ge 0$ and $r \ge 6$. The following problem is in* P*:*

$$\{G : \mathrm{cr}(G) \le c \wedge \chi(G) \le r\}.$$

Proof: We show that $\mathcal{G} = \{G : \mathrm{cr}(G) \le c\}$ is an awesome set and then apply Theorem 2.5. By Theorem 4.1, $\mathcal{G} \in P$. Clearly if $\mathrm{cr}(G) \le c$ and H is a subgraph of G, then $\mathrm{cr}(H) \le c$. By Theorem 4.4 we can take $\alpha = 3$ and $\beta = c$. ∎

4.3 When is the Problem Trivial?

There is no known analog of Theorem 3.4 for the crossing number. Hence we do not know the exact cutoff for when a graph coloring problems becomes trivial.

The crossing number of K_n is not known; however, Harary and Hill [8][1], Saaty [20], and Guy [7] have independently made the following conjecture:

[1]Mohar [15] claims that Hill made the conjecture in the 1950's

Conjecture 4.6 $\mathrm{cr}(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$.

Guy proved the conjecture for $1 \le n \le 10$. Pan and Richter [18] proved it for $n = 11$ and $n = 12$. Hence we have the following table

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\mathrm{cr}(K_n)$	0	0	0	0	0	1	3	9	18	36	60	100	150

At an AMS special session in Chicago in October 2007, Albertson made the following conjecture. We include both the conjecture and its contrapositive.

Conjecture 4.7

$$\chi(G) \ge r \implies \mathrm{cr}(G) \ge \mathrm{cr}(K_r).$$

The contrapositive of the conjecture is:

$$\mathrm{cr}(G) \le \mathrm{cr}(K_r) - 1 \implies \chi(G) \le r - 1.$$

Since $\mathrm{cr}(K_0) = \mathrm{cr}(K_1) = \mathrm{cr}(K_2) = \mathrm{cr}(K_3) = \mathrm{cr}(K_4) = 0$ *the conjecture is vacuously true for* $0 \le r \le 4$. *Hence we only consider the conjecture when* $r \ge 5$.

Albertson's conjecture has been proven for $5 \le r \le 16$. As noted above, for $5 \le r \le 12$, we know the value of $\mathrm{cr}(K_r)$. We combine the known values of $\mathrm{cr}(K_r)$ with the known cases of the conjecture to form the following table.

r	statement	Who proved it
5	$\mathrm{cr}(G) \le 0 \implies \chi(G) \le 4$	This is the 4-color theorem
6	$\mathrm{cr}(G) \le 2 \implies \chi(G) \le 5$	Oporowski and Zhao [17]
7	$\mathrm{cr}(G) \le 9 \implies \chi(G) \le 6$	Albertson, Cranston, Fox [1]
8	$\mathrm{cr}(G) \le 18 \implies \chi(G) \le 7$	Albertson, Cranston, Fox [1]
9	$\mathrm{cr}(G) \le 36 \implies \chi(G) \le 8$	Albertson, Cranston, Fox [1]
10	$\mathrm{cr}(G) \le 60 \implies \chi(G) \le 9$	Albertson, Cranston, Fox [1]
11	$\mathrm{cr}(G) \le 100 \implies \chi(G) \le 10$	Albertson, Cranston, Fox [1]
12	$\mathrm{cr}(G) \le 150 \implies \chi(G) \le 11$	Albertson, Cranston, Fox [1]
13	$\mathrm{cr}(G) \le \mathrm{cr}(K_{13}) \implies \chi(G) \le 12$	Barát and Tóth [4]
14	$\mathrm{cr}(G) \le \mathrm{cr}(K_{14}) \implies \chi(G) \le 13$	Barát and Tóth [4]
15	$\mathrm{cr}(G) \le \mathrm{cr}(K_{15}) \implies \chi(G) \le 14$	Barát and Tóth [4]
16	$\mathrm{cr}(G) \le \mathrm{cr}(K_{16}) \implies \chi(G) \le 15$	Barát and Tóth [4]

4.4 Summary of What is Known for Bounded Crossing Number

We summarize the complexity of the set

$$\mathrm{COL}_{c,r} = \{G : \mathrm{g}(G) \le c \wedge \chi(G) \le r\}.$$

1. If $r = 2$ and $c \ge 0$ then $\mathrm{COL}_{c,r}$ is in P. (Theorem 1.4.2)
2. If $r = 3$ and $c \ge 0$ then $\mathrm{COL}_{c,r}$ is NP-complete. (Theorem 1.4.3)
3. If $r = 4$ and $c = 0$ then $\mathrm{COL}_{c,r}$ is trivial. (Theorem 1.4.4)
4. If $r = 4$ and $c \ge 1$ then the complexity of $\mathrm{COL}_{c,r}$ is open.
5. If $r = 5$ and $c \ge 0$ then $\mathrm{COL}_{c,r}$ is in P. (Theorem 4.3)
6. If $r \ge 6$ and $c \ge 0$ then $\mathrm{COL}_{c,r} \in \mathrm{P}$, but see next four points. (Theorem 4.5)
7. If $c \in \{1, 2\}$ and $r = 5$ then $\mathrm{COL}_{c,r}$ is trivial.
8. If $c \in \{3, \ldots, 9\}$ and $r = 6$ then $\mathrm{COL}_{c,r}$ is trivial.
9. If $c \in \{10, \ldots, 18\}$ and $r = 7$ then $\mathrm{COL}_{c,r}$ is trivial.
10. More results like the last three can be derived from the table.

5 Open Problems

1. The problems we stated were in P used Theorem 2.2, Theorem 4.3, or Theorem 3.3. The complete proofs are difficult. Are there easier proofs? Are there more efficient algorithms?
2. Let $r = 4$ and $g \ge 1$. What is the complexity of

$$\{G : \mathrm{g}(G) \le g \wedge \chi(G) \le r\}?$$

3. Let $r = 4$ and $c \ge 1$. What is the complexity of

$$\{G : \mathrm{cr}(G) \le c \wedge \chi(G) \le r\}?$$

4. Determine for which c, r the set

$$\{G : \mathrm{cr}(G) \leq c \wedge \chi(G) \leq r\}$$

is trivial.

5. Theorem 2.5 applies to classes of graphs with $m \leq \frac{10n}{3} + O(1)$; however, we only applied it to classes of graphs with $m \leq 3m + O(1)$. Find an interesting class of graphs with $m \leq \alpha n + \beta$ where $3 < \alpha < \frac{10}{3}$ that we can apply the theorem to.

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