Open Problems Column
Edited by William Gasarch
This Issue's Column!

Juris Hartmanis, one of the founders of modern complexity theory, passed away on July 29, 2022 at the age of 94. This column is a tribute to him. It is

Open Problems by or Inspired by Juris Hartmanis

Request for Columns!

I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be (1) broad or narrow or anywhere in between, and (2) really important or really unimportant or anywhere inbetween.

Open Problems by or Inspired by Juris Hartmanis

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1 Introduction

This column is a collection of open problem that were either by or inspired by Juris Hartmanis. There are many authors.

2 Extremely Sparse Sets

by Eric Allender

In a paper by Hartmanis, Immerman, Sewelson [HIS85] they raised the following question:

If determinism and nondeterminism coincide for doubly-exponential time, can there be extremely sparse sets in NP−P?

They proved the answer was **no**. But there was a bug in the proof [All91]. In fact Allender showed that there exists an oracle where the answer is **yes**. Hence techniques that relativize will not suffice to resolve the question.

In the mean time, some non-relativizing techniques have been developed, as well as investigations into the limitations of those techniques. This suggests three possible directions:

1. Prove that the answer is **no**, though this will require techniques that do not relativize.
2. Prove that the result algebraizes, as defined by Aaronson and Wigderson [AW09a] and hence may be hard to resolve.
3. Prove that the answer is **yes**. We doubt such a proof will be found soon.

3 Algebraic and Transcendental Numbers

3.1 The Complexity of Algebraic Numbers

by William Gasarch

In Hartmanis & Stearns's classic paper [HS65] they defined DTIME($T(n)$). This is that part of the paper people usually point to. We will point to a different part: the complexity of real numbers.

Let $\alpha \in \mathsf{R}$. We want to know the complexity of α. We say $\alpha \in$ DTIME($T(n)$) if there is a Turing machine that will, on input an empty tape, run forever and print the first n digits of α in time $O(T(n))$.

A number is *algebraic* if it is the root of some polynomial in $\mathsf{Z}[x]$. A number is *transcendental* if it is not algebraic. We only deal with real numbers in this section (and the next).

Hartmanis & Stearns did the following:

- Observed that every rational is in DTIME(n).
- Proved that every algebraic number is in DTIME(n^2).
- Proved that there exists a transcendental number that is in DTIME(n).

They asked the question

Is there an algebraic number that requires $\Omega(n^2)$ time?

They noted that if there is then there would be an algebraic number that is more complicated than a transcendental number.

We list their question and a few more:

1. Does there exist an algebraic number that requires DTIME(n^2)? DTIME($n^{2-\delta}$) for some $0 < \delta < 1$?
2. Is there any relation between the degree of an algebraic number (the min degree of a polynomial over Z that it satisfies) and its complexity? We suspect not.
3. By an easy diagonalization one can show the following: for all computable $T(n)$, there exists an α that requires time $T(n)$. The α is unnatural. Find concrete examples for such with $T(n) = n^2$ or $T(n) = n^3$ or whatever your favorite function is.
4. There is a vast literature on computing either the first n digits (or bits) of π or just the nth bit. This is another direction to go.

There has been little progress on their original question or our additions; however, Freivalds [Fre12] has a survey of the work that HAS been done.

3.2 Real Time Computability and Transcendental Numbers

by Jin-Yi Cai

A real number r is said to be real-time computable if there exists a multitape Turing machine (TM) which on blank input, prints the binary expansion of r and gives the first n bits in $O(n)$ steps. P. Fischer, A. Meyer and A. Rosenberg [FMR70] showed that this is equivalent to the following: Some TM on input 1^n outputs the first n bits in $O(n)$ steps. The trick is to run two parallel subroutines which take turns to print the first $n = 2^k$ bits for increasing k; when one prints (part of) the first 2^k bits, the other prepares the (second half of) 2^{k+1} bits.

Every rational number is real-time computable, as its expansion is eventually periodic—one can use a finite automaton. A more interesting example is the (decimal) Champernowne's number

$$C_{10} = 0.1234567891011121314151617181920212 2\ldots.$$

A TM M can print the decimal values $k = 1, 2, 3, \ldots$ successively as follows: M uses two tapes holding a counter k on one tape and its head scanning left to right, and a second tape holding $k - 1$, to be updated to $k + 1$ with its head going right to left. More interestingly, the following numbers are real-time computable:

$$\sum_{n\geq 1} \frac{1}{2^{n^2}}, \quad \sum_{n\geq 1} \frac{1}{2^{n^3}}, \quad \sum_{n\geq 1} \frac{1}{2^{n!}}.$$

These numbers are real-time computable because their nonzero bits occur very sparsely. This latter property implies that they have very good binary rational approximations.

A number is algebraic if it is a root of a polynomial in $\mathbb{Z}[x]$. It is transcendental if it is not algebraic. Proving specific numbers transcendental is a hard problem, and historically it is intimately related to how close a number can be approximated by rational numbers. Liouville pioneered this line of inquiry and showed that a non-rational algebraic number cannot have rational approximations that are too close. This was used by Liouville, in the 1850's, to prove that transcendental numbers exist, and numbers such as $\sum_{n\geq 1} \frac{1}{2^{n!}}$ are transcendental. This method led to the proofs by (1) Hermite, in 1873, that e is transcendental, and (2) Lindemann, in 1882, that π is transcendental. The latter of course was the negative solution to the ancient Greek problem of squaring the circle. Mahler, in 1937, proved that C_{10} is transcendental. The transcendence of $\sum_{n\geq 1} \frac{1}{2^{n^2}}$ was only proved in 1996, by Bertrand [Ber97] and Duverney et al. [DNNS96] (independently). The result required deep results about algebraic independence of values of Eisenstein's series. The transcendence of $\sum_{n\geq 1} \frac{1}{2^{n^3}}$ is still open. Contrast this with Cantor's proof that transcendental numbers exist because the algebraic numbers are countable.

The question on how well an algebraic number can be rationally approximated culminated in Roth's theorem, from 1955, that a non-rational algebraic number cannot be approximated better than the order $c_\epsilon/q^{2+\epsilon}$ by rational numbers of the form $\frac{p}{q}$, for every $\epsilon > 0$. (Every real number has such approximations to the order c/q^2.)

In their Turing award-winning paper introducing time complexity classes in 1965, Hartmanis and Stearns proposed the following open problem:

Is every real-time computable number either rational or transcendental?

This has become known as the Hartmanis-Stearns Conjecture. If it is true, it would imply a deep connection between transcendental numbers and computational complexity.

The Hartmanis-Stearns Conjecture is true for finite automata (FA), and in fact true for deterministic pushdown automata (see the paper by Adamczewski-Cassigne-Gonidec [ACG20] and the references therein).

A sequence is b-automatic if there is a FA, when given n in base b expansion, produces the n-th term at the end. They prove that irrational automatic numbers are transcendental. Their work uses a generalization of Roth's theorem, and introduced a new criterion for transcendence. On the other hand, the work by Bailey-J. Borwein-Plouffe [BBP97] (see also J. Borwein & J. Borwein [BB87]) show that some transcendental numbers have surprisingly fast computable approximations. E.g., formulae such as

$$\pi = \sum_{k \geq 0} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right)$$

can lead to quasi-linear time computation of the bits of π. Yap [Yap10] (also see [LR13, Chapter 31] or [LR10]) showed that the digits of π can be computed in logspace. Allender et al. [ABDPar] improved this by showing that the digits of π can be computed in TC_0.

Concerning transcendental numbers, mystery persists. E.g., it is still unknown whether $e + \pi, e\pi$ and π^e are transcendental, but e^π is. Hilbert's 7th problem asks for a proof that if $\alpha \neq 0, 1$ and non-rational β are both algebraic, then α^β is transcendental. This was proved by Gel'fond and Schneider (independently in 1934). It was generalized to the famous Baker's theorem (1966): If $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$ are nonzero algebraic numbers such that there are no non-trivial relations of the form $\alpha_1^{n_1} \cdots \alpha_k^{n_k} = 1$, then $\log(\alpha_1^{\beta_1} \cdots \alpha_k^{\beta_k})$ is transcendental. Relations of the form above also have connections to complexity theory lately, e.g., such relations—called lattice conditions—are used in the proof of dichotomy theorems for counting problems by Cai, Guo and Williams [CGW16].

4 Sizes of DPDA's and PDA's

by William Gasarch

Convention 4.1 A *device* will either be a recognizer (e.g., a DFA) or a generator (e.g., a regular expression). We will use $\mathcal{M}$ to denote a set of devices (e.g., DFAs). We will refer to an element of $\mathcal{M}$ as an $\mathcal{M}$-device. If P is an $\mathcal{M}$-device then let $L(P)$ be the language recognized or generated by P. Let $L(\mathcal{M}) = \{L(P) : P \in \mathcal{M}\}$.

Def 4.2 Let $\mathcal{M}$ and $\mathcal{M}'$ be two sets of devices such that $L(\mathcal{M}) \subseteq L(\mathcal{M}')$. (e.g., DFAs and DPDAs). A *bounding function for* $(\mathcal{M}, \mathcal{M}')$ is a function f such that for all $A \in L(\mathcal{M})$, if $A \in L(\mathcal{M}')$ via an $\mathcal{M}'$-device of size n, then $A \in L(\mathcal{M})$ via an $\mathcal{M}$-device of size $\le f(n)$.

Valiant [Val76], and later Hartmanis [Har80] with an easier proof, showed the following:

Theorem 4.3 *If f is a bounding function for* $(\mathrm{DPDA}, \mathrm{PDA})$*, then* $\mathrm{HALT} \le_{\mathrm{T}} f$.

Theorem 4.3 shows that there is a large difference between sizes of DPDA's and PDA's for the same language. However, the theorem does not give us a concrete example of a DPDA language which has a much smaller PDA than DPDA.

Beigel & Gasarch [BG16], drawing heavily on Filmus [Fil11], showed the following.

Def 4.4

1. $W_n = \{ww : |w| = n\}$.
2. $A_n = \overline{W_n}$.

Theorem 4.5 *For almost all n:*

1. *There is a PDA of size $O(n)$ for A_n.*
2. *Any DPDA that recognizes A_n requires size $\ge 2^{2^{\Omega(n)}}$.*

Hence we have a double-exp gap between DPDA's and PDA's. Can we do better? Yes and No. Beigel & Gasarch [BG16] proved the following, drawing heavily on Hartmanis [Har80].

Def 4.6 INF is the set of all Turing machines (actually their indices) M such that M halts on an infinite number of inputs.

Theorem 4.7 *Let f be such that* $\mathrm{INF} \not\le_{\mathrm{T}} f$. *For infinitely many n there exists a language B_n such that:*

1. *There is a PDA of size n that accepts B_n.*
2. *Any DPDA that accepts B_n is of size $\ge f(n)$.*

(The set B_n is contrived. It involves Turing machines.)

Realize that for any computable f, including (say) Ackerman's function, $\mathrm{INF} \not\le_{\mathrm{T}} f$. So Theorem 4.7 *seems* like a much bigger separation than Theorem 4.5. But wait! Theorem 4.5 is an almost-all-n result, whereas Theorem 4.7 is an infinitely-many-n result. So they are actually incomparable. We would like to have the best of both worlds.

Open Question: For some f such that $2^{2^n} \ll f(n)$ show the following:

For almost all n there exists a language C_n such that:

1. There is a PDA of size $O(n)$ for C_n.
2. Any DPDA that recognizes C_n requires size $\ge f(n)$.
3. Bonus points if C_n is not constructed by diagonalization and does not involve Turing machines.

5 The Berman-Hartmanis Conjecture

5.1 The Berman-Hartmanis Isomorphism Conjecture: Origins

by Stuart A. Kurtz and James S. Royer

The Berman-Hartmanis Isomorphism Conjecture [BH77] posits that all **NP**-complete sets are isomorphic under polynomial-time isomorphisms.[1] As evidence for their conjecture, they showed that SAT is paddable,[2] and isomorphic to all paddable **NP**-complete sets, including all of the then-known "natural" **NP**-complete sets.

Analogs to the Berman-Hartmanis Conjecture include the Cantor-Bernstein Theorem of set theory[3] and Myhill's Theorem of computability theory[4], which are both true, and indeed the proof of Myhill's theorem can be adapted to the complexity-theoretic context, albeit at the cost of a slightly weaker result, which we'll call
Theorem A: If $f : A \to B$ and $g : B \to A$ are polynomial-time computable, 1-1, *invertible, and length-increasing*, then A and B are polynomial-time isomorphic.

Deborah Joseph and Paul Young observed that the gap between Theorem A and the Isomorphism Conjecture is uncomfortably large. In particular, if f is a polynomial-time 1-way function (in this case, meaning a function such that the only polynomial-time computable sets contained in its range are sparse), then $f(\mathrm{SAT})$ will be **NP**-complete, but non-paddable, and so not isomorphic to SAT, contracting the conjecture.

The Isomorphism Conjecture and related work were a strong drivers in the emergence of *structural complexity*, which emphasized reductions and degree structure as an approach to studying complexity theory. There are oracles known relative to which the conjecture holds, relative to which it fails, and (perhaps most surprisingly) relative to which it holds, even though certain sorts of 1-way functions exist.

5.2 The Cylinder Conjecture

by Lance Fortnow

Suppose we wanted to find a counterexample to the Berman-Hartmanis Isomorphism Conjecture [BH77]. Consider the following family of languages:

$$\{f(\mathrm{SAT})\colon f : \Sigma^* \to \Sigma^*\} ,$$

where $f(\mathrm{SAT}) = \{f(x) : x \in \mathrm{SAT}\}$.

If f is polynomial-time computable, injective and length non-decreasing then $f(\mathrm{SAT})$ is NP-complete. For the rest of this section we will limit ourselves to f that have these properties.

[1] Polynomial-time computable 1-1 and onto functions whose inverse is also polynomial-time computable

[2] There is a polynomial-time computable and invertible pairing function p such that $p(x, y) \in A \iff x \in A$.

[3] If $f : A \to B$ and $g : B \to A$ are 1-1, then there exists a bijective $h : A \to B$.

[4] If A and B are c.e.-complete under computable 1-reductions, then A and B are computably isomorphic.

The idea is to find an f *complicated enough* so that $f(\mathrm{SAT})$ is not isomorphic to SAT. Deborah Joseph and Paul Young [JY85] first considered this approach in their study of k-creative sets.

In 1995 Stuart Kurtz, Steve Mahaney and Jim Royer [KMR95] define the notion of a scrambling function, a function f such that the range of f does not contain a non-empty paddable language, i.e, where there is a polynomial-time computable and invertible injective function g such that for all strings x and y, x is in L if and only if $g(x, y)$ is in L. They show

1. For any scrambling function f, $f(\mathrm{SAT})$ is not isomorphic to SAT.

2. Relative to a random oracle, scrambling functions exist.

As an immediate corollary, the Berman-Hartmanis conjecture fails relative to a random oracle.

Based on this work, the Berman-Hartmanis conjecture was generally considered likely false as scrambling or other similar functions seemed reasonably likely to exist in the unrelativized world. Or so we thought until 2009 when Manindra Agrawal and Osamu Watanabe [AW09b] showed that for the known one-way function candidates f, $f(\mathrm{SAT})$ is isomorphic to SAT, at least non-uniformly.

Intuitively, Agrawal and Watanabe show that if f has an easy cylinder than $f(\mathrm{SAT})$ is isomorphic to SAT. A cylinder is a way to embed Σ^* into an invertible range of f, informally, two easy-to-compute functions e and g such that for all x, $g(f(e(x))) = x$. The formal definition they need is a bit more technical [AW09b, Definition 3]. Agrawal and Watanabe made the following conjecture:

Conjecture (Cylinder Conjecture) If f is easy to compute then f has a non-uniform easy cylinder.

Shortly after Agrawal and Watanabe made their conjecture, Oded Goldreich published a potential counterexample one-way function based on expander graphs [Gol11]. Goldreich's function composes a function with itself several times depending on the input length. Agrawal and Watanabe counter that if one iterates a function with an easy cylinder in this manner, the iterated function should also have an easy cylinder, though they can't prove this point.

The cylinder conjecture remains open and is likely the key to whether the isomorphism conjecture is true in the unrelativized world. While it can't be settled with a relativizing proof, more evidence of the conjecture holding such as a proof that Goldreich's function has an easy cylinder, or a more convincing counterexample would help us better conjecture whether the isomorphism conjecture may be true.

5.3 The BH Conjecture with Weaker Reductions

by Neil Immerman

Hartmanis and his student, Len Berman, considered the question of whether all NP-complete problems are the same or if they vary. Myhill had proved that all r.e. complete problems are recursively isomorphic [Myh55]. Berman and Hartmanis made the following conjecture:
Berman-Hartmanis Isomorphism Conjecture [BH77]: If A, B are NP-complete via ptime many-one reductions, then A and B are ptime isomorphic ($A \cong_p B$).

The Berman-Hartmanis Isomorphism Conjecture remains open. In particular, it implies that P≠NP, but even the weaker conjecture — If P≠NP Then the Berman-Hartmanis Isomorphism Conjecture holds — is open.

In the late 1970's, when Steve Mahaney and I were his grad students, Hartmanis was very interested in proving structural properties of NP-complete sets. Mahaney succeeded in doing just that, proving
Mahaney's Theorem [Mah82]: If P≠NP then all NP-complete problems are dense.

Reductions

Hartmanis was also interested in the fact that complete problems seem to remain complete via surprisingly weak reductions. Initially, Cook proved that SAT is NP-complete via ptime Turing reductions [Coo71]. When Karp produced many other important NP-complete problems, he used ptime, many-one reductions [Kar72]. Jones showed that they stay complete via logspace reductions [Jon75]. Hartmanis, Immerman and Mahaney showed that one-way logspace reductions suffice [HIM78]; in this model the transducer reads its input once from left to right.

When I introduced Descriptive Complexity, I was pleased to see that first-order reductions — which are the natural way to translate one logical problem to another — preserve the completeness properties for all natural complete problems for all complexity classes [Imm99]. When I mentioned this to Mike Sipser, he pointed out that Valiant's projections are weaker and still preserve natural complete problems [Val82]. Projections are non-uniform reductions which perform no computations: the ith output bit is either always 0 or always 1, or it is a fixed bit, $b_{f(i)}$, of the input, or the negation of a fixed bit, $\neg b_{f(i)}$.

I was pleased to observe that there is a natural restriction of first-order reductions making them (first-order uniform) projections. I call these first-order projections (fops) and observe that natural problems stay complete via fops [Imm99]. A bonus is that fops are so well-behaved that we can prove an isomorphism theorem:
Fops Isomorphism Theorem [ABI97] For all important complexity classes $\mathcal{C}$, (this includes L, NL, P, NP, PSPACE, EXPTIME) every two problems complete for $\mathcal{C}$ via fops are first-order isomorphic.

I felt that this was a positive answer to the Berman-Hartmanis Conjecture, and Hartmanis agreed with me. Later, Agrawal strengthened our theorem to the following:
First-Order Isomorphism Theorem [Agr01] For all important complexity classes, $\mathcal{C}$, every two problems complete for $\mathcal{C}$ via first order reductions are first-order isomorphic.

Dichotomy Phenomenon

We are familiar with the fact that "natural" computational problems tend to be complete (and in fact complete via fops) for one of our favorite complexity classes. Furthermore, from the Fops Isomorphism Theorem, we can conclude that these natural problems are really a very small number of problems — only one for each of our favorite complexity classes.
Open Problem: Understand, explain and make use of this phenomenon.

A related and equally hard problem was frequently proposed by Hartmanis: separate complexity classes by proving that certain weak reductions do not exist. Here is one example. The problem REACH_d is the set of directed graphs of outdegree one having a path from s to t. A quantifier-free projection (qfp) is a fop that happens to be quantifier-free.
Theorem [Imm99] REACH_d is complete for DSPACE$[\log n]$ via qfps.
Corollary $(\mathrm{NP} = \mathrm{DSPACE}[\log n]) \quad \Leftrightarrow \quad \text{3-COLOR} \leq_{\mathrm{qfp}} \mathrm{REACH}_d$
Open problem: Develop techniques towards proving that there is no qfp from 3-COLOR to REACH_d.

6 Amplification/Magnification/Bootstrapping for SAT

by Ryan Williams

Juris and I discussed many problems, but one stands out for me as particularly prescient, given about 20 years of hindsight. We start with the following curious observation.
Theorem. *There exists a fixed constant c such that* $\mathrm{P} = \mathrm{NP}$ *if and only if* SAT *is in* $O(n^c)$ *time.*

The proof is trivial but nastily non-constructive. There are two cases. First, if $\mathrm{P} = \mathrm{NP}$ then SAT is in $O(n^k)$ time for some k, so we may set $c = k$. Second, if $\mathrm{P} \neq \mathrm{NP}$ then the statement is true for every c.
Open Problem: Find an *explicit* constant c for which the above theorem holds. (For example, does the theorem hold for $c = 10$?)

Juris told me a result like this would truly convince him that we've made progress on $\mathrm{P} \neq \mathrm{NP}$: from (say) an n^{10}-time lower bound for SAT, we could conclude a super-polynomial time lower bound. This came up while we were discussing Fortnow's journal paper on time-space tradeoffs for SAT [For00], and the possibility of improving the time lower bound beyond the golden ratio exponent established by Fortnow and Van Melkebeek in CCC'00 [FLvMV00], which I eventually did [Wil08].[5]

I had forgotten about this conversation with Juris until recently. At the time, it felt like an impossible problem to me. However, several years after our discussion, Allender and Koucky [AK10] released their paper on *Amplifying lower bounds by means of self-reducibility*, showing how $n^{1+\varepsilon}$-type TC^0 lower bounds on (for example) Formula Evaluation would imply $\mathrm{NC}^1 \neq \mathrm{TC}^0$ outright (you cannot compute Formula Evaluation on TC^0 circuits

[5]Sometimes when I'm feeling blue, I pull out an old email from Juris, documenting his response: *My very sincere congratulations!!! This is GREAT !!* May he rest in peace.

of *any* polynomial size). It is possible that he was also aware of Aravind Srinivasan's STOC 2000 paper [Sri00] showing that *weak-looking* time lower bounds on approximating CLIQUE would imply P $\neq$ NP (I was unaware of it, at the time).

There are now many *amplification* results of a similar flavor, sometimes also called *hardness magnification* or *bootstrapping*, where one shows that a fixed-polynomial lower bound for one problem implies a super-polynomial lower bound (for possibly a different problem), by taking advantage of problem structure [AAW10, LW10, OS18, OPS19, CILM18, MMW19, CT19, CMMW19, CJW19, Hir20, MP20, CJW20, Fu20, CHO$^+$22, CLY22]. (I have tried to be exhaustive, but there are many recent papers! I hope I didn't leave yours out.)

Maybe now, this open problem is not quite as impossible as it used to be. Variants of the problem are also just as interesting. Could it be that SAT is not solvable by an algorithm running in both cubic time and logspace if and only if NP $\neq$ LOGSPACE? That would partially explain why it seems so difficult to prove a super-quadratic time lower bound for SAT against logspace machines (for both the decision version of SAT, and the search version).

References

[AAW10] Eric Allender, Vikraman Arvind, and Fengming Wang. Uniform derandomization from pathetic lower bounds. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, 13th International Workshop, APPROX 2010, and 14th International Workshop, RANDOM 2010*, pages 380–393, 2010. https://link.springer.com/content/pdf/10.1007/978-3-642-15369-3_29.pdf.

[ABDPar] Eric Allender, Nikhil Balaji, Samir Datta, and Rameshwar Pratap. On the complexity of algebraic numbers, and the bit-complexity of straight-line programs. *Computability (The Journal of the Association Computability in Europe)*, 2022 (to appear). https://eccc.weizmann.ac.il/report/2022/053.

[ABI97] Eric Allender, José L. Balcázar, and Neil Immerman. A first-order isomorphism theorem. *SIAM Journal of Computing*, 26(2):557–567, 1997. https://doi.org/10.1137/S0097539794270236.

[ACG20] Boris Adamczewski, Julien Cassigne, and Marion Le Gonidec. On the computational complexity of algebraic numbers: The Hartmanis-Stearns problem revisited. *Transactions of the the American Mathematics Society*, 373:3085–3115, 2020. https://hal.archives-ouvertes.fr/hal-01254293/document.

[Agr01] Manindra Agrawal. The first-order isomorphism theorem. In Ramesh Hariharan, Madhavan Mukund, and V. Vinay, editors, *FST TCS 2001: Foundations of*

Software Technology and Theoretical Computer Science, 21st Conference, Bangalore, India, December 13-15, 2001, Proceedings, volume 2245 of *Lecture Notes in Computer Science*, pages 70–82. Springer, 2001.
https://doi.org/10.1007/3-540-45294-X_7.

[AK10] Eric Allender and Michal Koucký. Amplifying lower bounds by means of self-reducibility. *Journal of the ACM*, 57(3):14:1–14:36, 2010.
https://iuuk.mff.cuni.cz/~koucky/papers/gap.pdf.

[All91] Eric Allender. Limitations of the upward separation technique. *Math. Syst. Theory*, 24(1):53–67, 1991.
10.1007/BF02090390.

[AW09a] Scott Aaronson and Avi Wigderson. Algebraization: A new barrier to complexity theory. *ACM Transactions on Computing Theory*, 1, 2009.
https://www.scottaaronson.com/papers/alg.pdf.

[AW09b] Manindra Agrawal and Osamu Watanabe. One-way functions and the Berman-Hartmanis conjecture. In *Proceedings of the 24th Annual IEEE Conference on Computational Complexity, CCC 2009, Paris, France, 15-18 July 2009*, pages 194–202. IEEE Computer Society, 2009.
https://doi.org/10.1109/CCC.2009.17.

[BB87] Jonathan Borwein and Peter Borwein. *Pi and the AGM.* John Wiley and Sons, 1987.
https://doi.org/10.1016/S0022-0000(70)80012-5.

[BBP97] David Bailey, Peter Borwein, and Simon Plouffe. On the rapid computation of various polylogarithmic constants. *Mathematics of Computation*, pages 903–913, 1997.
https://www.davidhbailey.com/dhbpapers/digits.pdf.

[Ber97] Daniel Bertrand. Theta functions and transcendence. *Ramanjuan Journal*, pages 339–350, 1997.
https://link.springer.com/content/pdf/10.1023/A:1009749608672.pdf.

[BG16] Richard Beigel and William I. Gasarch. On the sizes of DPDAs, PDAs, LBAs. *Theor. Comput. Sci.*, 638:63–75, 2016.
https://doi.org/10.1016/j.tcs.2015.08.028.

[BH77] Leonard Berman and Juris Hartmanis. On isomorphism and density of NP and other complete sets. *SIAM Journal on Computing*, 6:305–322, 1977.
https://dl.acm.org/doi/10.1145/800113.803628.

[CGW16] Jin-Yi Cai, Heng Guo, and Tyson Williams. The complexity of counting edge colorings and a dichotomy for some higher domain holant problems. *Reseach Math Sciences*, 3(18), 2016.
https://link.springer.com/article/10.1186/s40687-016-0067-8.

[CHO+22] Lijie Chen, Shuichi Hirahara, Igor Carboni Oliveira, Ján Pich, Ninad Rajgopal, and Rahul Santhanam. Beyond natural proofs: Hardness magnification and locality. *Journal of ACM*, 69(4):25:1–25:49, 2022.
https://doi.org/10.1145/3538391.

[CILM18] Marco L. Carmosino, Russell Impagliazzo, Shachar Lovett, and Ivan Mihajlin. Hardness amplification for non-commutative arithmetic circuits. In *33rd Computational Complexity Conference, CCC 2018, June 22-24, 2018, San Diego, CA, USA*, pages 12:1–12:16, 2018.
https://drops.dagstuhl.de/opus/volltexte/2018/8877/pdf/LIPIcs-CCC-2018-12.pdf.

[CJW19] Lijie Chen, Ce Jin, and R. Ryan Williams. Hardness magnification for all sparse NP languages. In *IEEE Annual Symposium on Foundations of Computer Science, (FOCS)*, pages 1240–1255, 2019.
https://ieeexplore.ieee.org/document/8948681.

[CJW20] Lijie Chen, Ce Jin, and R. Ryan Williams. Sharp threshold results for computational complexity. In *ACM Symposium on Theory of Computing, (STOC)*, pages 1335–1348, 2020.
https://dl.acm.org/doi/pdf/10.1145/3357713.3384283.

[CLY22] Lijie Chen, Jiatu Li, and Tianqi Yang. Extremely efficient constructions of hash functions, with applications to hardness magnification and PRFs. In *Computational Complexity Conference, (CCC)*, pages 23:1–23:37, 2022.
https://drops.dagstuhl.de/opus/volltexte/2022/16585/pdf/LIPIcs-CCC-2022-23.pdf.

[CMMW19] Lijie Chen, Dylan M. McKay, Cody D. Murray, and R. Ryan Williams. Relations and equivalences between circuit lower bounds and Karp-Lipton theorems. In *Computational Complexity Conference (CCC)*, pages 30:1–30:21, 2019.
https://drops.dagstuhl.de/opus/volltexte/2019/10852/pdf/LIPIcs-CCC-2019-30.pdf.

[Coo71] Stephen Cook. The complexity of theorem proving procedures. In *Proceedings of the Third Annual ACM Symposium on the Theory of Computing,* Shaker Heights OH, pages 151–158, 1971.
https://www.inf.unibz.it/~calvanese/teaching/11-12-tc/material/cook-1971-NP-completeness-of-SAT.pdf.

[CT19] Lijie Chen and Roei Tell. Bootstrapping results for threshold circuits “just beyond” known lower bounds. In *ACM Symposium on Theory of Computing*, pages 34–41, 2019.
https://dl.acm.org/doi/10.1145/3313276.3316333.

[DNNS96] Daniel Duverney, Keij Nishiika, Kumiko Nishioka, and Iekata Shiokawa. Transcendence of Jacobi's theta series. *Proceedings of the Japanese Academy of Mathematical Science*, pages 202–203, 1996.
file:///C:/Users/BillLoc/Downloads/pjaa.72.202.pdf.

[Fil11] Yuval Filmus. Lower bounds for context-free grammars. *Information Processing Letters*, 111(18):895–898, 2011.
https://www.sciencedirect.com/science/article/pii/S0020019011001712.

[FLvMV00] Lance Fortnow, Richard J. Lipton, Dieter van Melkebeek, and Anastasios Viglas. Time-space lower bounds for satisfiability. *Journal of the ACM*, 52(6):835–865, 2005. Merger of FOCS'99 and CCC'00.
https://people.cs.uchicago.edu/~fortnow/papers/tst.pdf.

[FMR70] Patrick C. Fischer, Albert R. Meyer, and Arnold L. Rosenberg. Time-restricted sequence generation. *J. Comput. Syst. Sci.*, 4(1):50–73, 1970.
https://doi.org/10.1016/S0022-0000(70)80012-5.

[For00] Lance Fortnow. Time-space tradeoffs for satisfiability. *J. Comput. Syst. Sci.*, 60(2):337–353, 2000.
https://lance.fortnow.com/papers/files/npvsnl.pdf.

[Fre12] Rusins Freivalds. Hartmanis-Stearns conjecture on real time and transcendence. In Michael J. Dinneen, Bakhadyr Khoussainov, and André Nies, editors, *Computation, Physics and Beyond - International Workshop on Theoretical Computer Science, WTCS 2012, Dedicated to Cristian S. Calude on the Occasion of His 60th Birthday, Auckland, New Zealand, February 21-24, 2012, Revised Selected and Invited Papers*, volume 7160 of *Lecture Notes in Computer Science*, pages 105–119. Springer, 2012.
https://doi.org/10.1007/978-3-642-27654-5_9.

[Fu20] Bin Fu. Hardness of sparse sets and minimal circuit size problem. In *Computing and Combinatorics - 26th International Conference (COCOON)*, pages 484–495, 2020.
https://arxiv.org/abs/2003.00669.

[Gol11] Oded Goldreich. A candidate counterexample to the easy cylinders conjecture. In Oded Goldreich, editor, *Studies in Complexity and Cryptography. Miscellanea on the Interplay between Randomness and Computation - In Collaboration with*

Lidor Avigad, Mihir Bellare, Zvika Brakerski, Shafi Goldwasser, Shai Halevi, Tali Kaufman, Leonid Levin, Noam Nisan, Dana Ron, Madhu Sudan, Luca Trevisan, Salil Vadhan, Avi Wigderson, David Zuckerman, volume 6650 of *Lecture Notes in Computer Science*, pages 136–140. Springer, 2011.
`https://doi.org/10.1007/978-3-642-22670-0\_16`.

[Har80] Juris Hartmanis. On the succinctness of different representations of languages. *SIAM Journal on Computing*, 9(1):114–120, 1980.
`https://epubs.siam.org/doi/abs/10.1137/0209010`.

[HIM78] Juris Hartmanis, Neil Immerman, and Stephen R. Mahaney. One-way log-tape reductions. In *19th Annual Symposium on Foundations of Computer Science, Ann Arbor, Michigan, USA, 16-18 October 1978*, pages 65–72. IEEE Computer Society, 1978.
`https://doi.org/10.1109/SFCS.1978.31`.

[Hir20] Shuichi Hirahara. Non-disjoint promise problems from meta-computational view of pseudorandom generator constructions. In *Computational Complexity Conference (CCC)*, pages 20:1–20:47, 2020.
`https://drops.dagstuhl.de/opus/volltexte/2020/12572/`.

[HIS85] Juris Hartmanis, Neil Immerman, and Vivian Sewelson. Sparse sets in NP−P: EXPTIME versus NEXPTIME. *Information and Computation*, 65:158–181, 1985.
`https://dl.acm.org/doi/pdf/10.1145/800061.808769`.

[HS65] Juris Hartmanis and Richard E. Stearns. On the computational complexity of algorithms. *Transactions of the American Math Society*, 117:285–306, 1965.
`https://www.ams.org/journals/tran/1965-117-00/S0002-9947-1965-0170805-7/S0002-9947-1965-0170805-7.pdf`.

[Imm99] Neil Immerman. *Descriptive Complexity Theory*. Springer Verlag, New York, Heidelberg, Berlin, 1999.

[Jon75] Neil D. Jones. Space-bounded reducibility among combinatorial problems. *J. Comput. Syst. Sci.*, 11(1):68–85, 1975.
`https://doi.org/10.1016/S0022-0000(75)80050-X`.

[JY85] Deborah Joseph and Paul Young. Some remarks on witness functions for non-polynomial and noncomplete sets in NP. *Theor. Comput. Sci.*, 39:225–237, 1985.
`https://doi.org/10.1016/0304-3975(85)90140-9`.

[Kar72] Richard Karp. Reducibilities among combinatorial problems. In Ray Miller and J.W. Thatcher, editors, *Complexity of computer computations*, pages 85–103.

Plenum Press, 1972.
https://www.cs.umd.edu/~gasarch/BLOGPAPERS/Karp.pdf.

[KMR95] Stuart A. Kurtz, Stephen R. Mahaney, and James S. Royer. The isomorphism conjecture fails relative to a random oracle. *J. ACM*, 42(2):401–420, 1995.
https://doi.org/10.1145/201019.201030.

[LR10] Richard Lipton and Ken Regan. Making an algorithm an algorithm-BBP, 2010.
https://rjlipton.wpcomstaging.com/2010/07/14/making-an-algorithm-an-algorithm-bbp/.

[LR13] Richard Lipton and Kenneth Regan. *People, problems, and proof: essays from Godel's last letter: 2010.* Springer, New York, Heidelberg, Berlin, 2013.

[LW10] Richard J. Lipton and Ryan Williams. Amplifying circuit lower bounds against polynomial time, with applications. *Computational Complexity*, 22(2):311–343, 2013. Conference version in CCC'10.
https://people.csail.mit.edu/rrw/qbf-nc-journal-rev.pdf.

[Mah82] Steven Mahaney. Sparse complete sets for NP: Solution to a conjecture of Berman and Hartmanis. *Journal of Computer and System Sciences*, 25:130–143, 1982.
https://ieeexplore.ieee.org/document/4567805.

[MMW19] Dylan M. McKay, Cody D. Murray, and R. Ryan Williams. Weak lower bounds on resource-bounded compression imply strong separations of complexity classes. In *ACM Symposium on Theory of Computing (STOC)*, pages 1215–1225, 2019.
https://dl.acm.org/doi/10.1145/3313276.3316396.

[MP20] Moritz Müller and Ján Pich. Feasibly constructive proofs of succinct weak circuit lower bounds. *Ann. Pure Appl. Log.*, 171(2), 2020.
https://doi.org/10.1016/j.apal.2019.102735.

[Myh55] John Myhill. Creative sets. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 1:97–108, 1955.

[OPS19] Igor Carboni Oliveira, Ján Pich, and Rahul Santhanam. Hardness magnification near state-of-the-art lower bounds. In *Computational Complexity Conference (CCC)*, pages 27:1–27:29, 2019.
https://theoryofcomputing.org/articles/v017a011/v017a011.pdf.

[OS18] Igor Carboni Oliveira and Rahul Santhanam. Hardness magnification for natural problems. In *IEEE Annual Symposium on Foundations of Computer Science*

(FOCS), pages 65–76, 2018.
https://ieeexplore.ieee.org/document/8555094.

[Sri00] Aravind Srinivasan. The value of strong inapproximability results for clique. In *ACM Symposium on Theory of Computing (STOC)*, pages 144–152, 2000.
https://dl.acm.org/doi/10.1145/335305.335322.

[Val76] Leslie G. Valiant. A note on the succinctness of descriptions of deterministic languages. *Inf. Control.*, 32(2):139–145, 1976.
https://doi.org/10.1016/S0019-9958(76)90173-X.

[Val82] Leslie Valiant. Reduciblity by algebraic projections. *L' Enseignement mathematique*, T. XXVIII:253–268, 1982.

[Wil08] Ryan Williams. Time space tradeoffs for counting NP solutions modulo integers. In *Computational Complexity*, pages 179–219, New York, 2008. IEEE.
http://www.stanford.edu/~rrwill/projects.html.

[Yap10] Chee Yap. π is in log space, 2010.
https://cs.nyu.edu/exact/doc/pi-log.pdf.