Open Problems Column
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This Issues Column! This issue's Open Problem Column is by William Gasarch and is *Placing Pennies on a Chessboard to Obtain Distinct Distances*.

Request for Columns! I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be (1) broad or narrow or anywhere inbetween, and (2) really important or really unimportant or anywhere inbetween.

Placing Pennies on a Chessboard to Obtain Distinct Distances
By William Gasarch

1 Introduction

The problem I will discuss I found out about at the AMS-MAA joint meeting in Boston in 2023. It seems to be due to Matt Parker; however, I have not been able to find a reference.

Convention 1.1 Throughout this paper (1) an $n \times n$ chessboard has all squares 1×1, (2) pennies have diameter 1, (3) the distance between two pennies is the distance between their centers.

Def 1.2 Let $k, n \in \mathbb{N}^+$ with $k \leq n$.

1. (k, n) is *placeable* if there is a way to place k pennies on an $n \times n$ chessboard such that all the pairs of distances between pennies are distinct.
2. n is *placeable* if (n, n) is placeable.

Matt Parker asked the following:

Which n are placeable?

Oscar Cunningham, in his blog here, solved the problem with help from Gal Holowitz. We state what he claims and give comments on it:

1. Oscar Cunningham used a computer search and found that, for $3 \leq n \leq 7$, n is placeable. This was also shown by Erdős and Guy [1] (see Section 4 for more on that paper) who we suspect did not use a computer. We give the placements of Erdős and Guy in the appendix.
2. Oscar Cunningham used a computer search to show that, for $8 \leq n \leq 11$, n is not placeable. Gal Holowitz used a computer search to show that, for $12 \leq n \leq 14$, n is not placeable.

3. Oscar Cunningham states that

   But for the $n = 15$ case their code is still running!

   That was written in 2020 so one presumes the code has stopped running. In any case, the $n = 15$ case did not seem amenable to a computer search. Oscar Cunningham then gave an elegant human-readable proof that 15 is not placeable. We present that proof in this paper.

4. Oscar Cunningham says that for all $n \geq 16$

   > *it is trivial that there are no solutions (what we call placements) because the number of pairs of counters (what we call pennies) is greater than the number of possible distances on the grid.*

   Despite this being called *trivial*, we provide a proof for completeness.

2 15 is NOT Placeable

The proof in this section is due to Oscar Cunningham and was on his blog here.

Def 2.1 Let $f(n)$ be the number of numbers that can be written as the sum of 2 squares in at least 2 ways using numbers from $\{0, \ldots, n-1\}$.

Lemma 2.2 *The number of distances between squares on the $n \times n$ chessboard is*

$$\leq \frac{n(n-1)}{2} + n - 1 - f(n).$$

Proof: The set of distances between squares is the set of distances from the left bottom square (LBS) to all of the other squares, minus repeats. First look at the distance from the LBS to the top right square. Then from the LBS to the two squares that are furthest away in the second-to-top row. Etc. This is $1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$. Then we add in all of the squares in the bottom row except the LBS. That's $n - 1$. We then subtract the number of repeats which is $\geq f(n)$. Hence the number of distances is

$$\leq 1 + 2 + 3 + \cdots + (n-1) + (n-1) - f(n) = \frac{n(n-1)}{2} + n - 1 - f(n)$$

∎

Theorem 2.3 *15 is not placeable.*

Proof:

$f(15) \ge 14$. We list 14 numbers that can be written as a^2+b^2 in at least 2 ways with $a,b \in \{0,\ldots,14\}$. It turns out that $f(15)=14$ though we do not need that and do not prove it.

$25 = 0^2+5^2 = 3^2+4^2$
$50 = 1^2+7^2 = 5^2+5^2$
$65 = 1^2+8^2 = 4^2+7^2$
$85 = 2^2+9^2 = 6^2+7^2$
$100 = 0^2+10^2 = 6^2+8^2$
$125 = 2^2+11^2 = 5^2+10^2$
$130 = 3^2+11^2 = 7^2+9^2$
$145 = 1^2+12^2 = 8^2+9^2$
$169 = 0^2+13^2 = 5^2+12^2$
$170 = 1^2+13^2 = 7^2+11^2$
$185 = 4^2+13^2 = 8^2+11^2$
$200 = 10^2+10^2 = 2^2+14^2$
$205 = 3^2+14^2 = 6^2+13^2$
$221 = 10^2+11^2 = 5^2+14^2$

Since $f(15) \ge 14$, by Lemma 2.2 the number of possible distances is

$$\le \frac{15\times 14}{2} + 14 - 14 = \frac{15\times 14}{2} = \binom{15}{2}.$$

AH-HA! To place 15 pennies you need to achieve $\binom{15}{2}$ distances. Hence EVERY distance must appear.

All of the distances are of the form $\sqrt{a^2+b^2}$ where $1 \le a \le b \le 14$. We list here the top 6 distances.

1. $\sqrt{14^2+14^2} = \sqrt{392}$.
2. $\sqrt{13^2+14^2} = \sqrt{365}$.
3. $\sqrt{12^2+14^2} = \sqrt{340}$.
4. $\sqrt{13^2+13^2} = \sqrt{338}$.
5. $\sqrt{11^2+14^2} = \sqrt{317}$.
6. $\sqrt{12^2+13^2} = \sqrt{313}$.

We try to place pennies to get these distances and show that we cannot.

The largest distance, $\sqrt{392}$, can only be achieved by having two pennies in opposite diagonal corners. Hence we can assume that p_1 and p_2 are as in Figure 1. We use X to indicate spots where no penny can go since they are equidistant from p_1 and p_2.

p_1														X
													X	
												X		
											X			
										X				
									X					
								X						
							X							
						X								
					X									
				X										
			X											
		X												
	X													
X														p_2

Figure 1: Placement of p_1 and p_2

1. $d(p_1, p_2) = \sqrt{392}$

The second largest distance, $\sqrt{13^2+14^2} = \sqrt{365}$ can only be achieved by going from a corner to the space next to the opposite diagonal corner . Given where we place p_1 and p_2 the only way to achieve this is to place p_3 as in Figure 2. We once again place X's in places where no penny can go.

p_1	p_3	X												X
X	X												X	
												X		
											X			
										X				
									X					
								X						
							X							
						X								
					X									
				X										
			X											
		X												
	X													X
X													X	p_2

Figure 2: Placement of p_1, p_2, p_3

1. $d(p_1, p_2) = \sqrt{392}$
2. $d(p_1, p_3) = 1$
3. $d(p_2, p_3) = \sqrt{365}$

The third largest distance, $\sqrt{12^2+14^2} = \sqrt{340}$ can only be achieved by going from a corner to two away from the diagonally opposite corner (pennies p_2 and p_5 in Figure 4). Given how we placed p_1, p_2, p_3 there are 3 ways to place a penny to achieve this. This is not what we want! We want there to be only one way! Hence we put off getting this distance for now.

The fourth largest distance, $\sqrt{13^2+13^2} = \sqrt{338}$, is from a corner to the square that is diagonally next to the diagonally opposite corner. Given where we placed p_1, p_2, p_3, the only way to achieve this is placing p_4 as in Figure 3. The X's are where we cannot place pennies.

1. The new X that is diagonally to the right of p_3 is $\sqrt{2}$ away from p_3.
2. The new X's that are on the diagonal right above the main diagonal are equidistant from p_1 and p_4.
3. The new X's in the bottom right corner are all either 1 or $\sqrt{2}$ away from p_4.

p_1	p_3	X											X	X
X	X	X										X	X	
											X	X		
										X	X			
									X	X				
								X	X					
							X	X						
						X	X							
					X	X								
				X	X									
			X	X										
		X	X											
	X	X										X	X	X
X	X											X	p_4	X
X												X	X	p_2

Figure 3: Placement of p_1, p_2, p_3, p_4

1. $d(p_1, p_2) = \sqrt{392}$
2. $d(p_1, p_3) = 1$
3. $d(p_1, p_4) = \sqrt{338}$
4. $d(p_2, p_3) = \sqrt{365}$

5. $d(p_2, p_4) = \sqrt{2}$

6. $d(p_3, p_4) = \sqrt{313}$

We now place the third largest distance, $\sqrt{14^2 + 12^2} = \sqrt{340}$. As noted earlier, this can only be achieved by going from a corner to two away from the diagonally opposite corner. Given how we place p_1, p_2, p_3, p_4 there we are forced to place p_5 as in Figure 4. We leave the justification for the new X's to the reader.

p_1	p_3	X											X	X
X	X	X										X	X	
p_5	X										X	X		
										X	X			
									X	X				
								X	X					
							X	X						
						X	X							
					X	X								
				X	X									
			X	X										
		X	X											
	X	X										X	X	X
X	X											X	p_4	X
X												X	X	p_2

Figure 4: Placement of p_1, p_2, p_3, p_4, p_5

1. $d(p_1, p_2) = \sqrt{392}$

2. $d(p_1, p_3) = 1$

3. $d(p_1, p_4) = \sqrt{338}$

4. $d(p_1, p_5) = 2$

5. $d(p_2, p_3) = \sqrt{365}$

6. $d(p_2, p_4) = \sqrt{2}$

7. $d(p_2, p_5) = \sqrt{340}$

8. $d(p_3, p_4) = \sqrt{313}$

p_1	p_3	X	p_6^2										X	X
X	X	X										X	X	
p_5	X										X	X		
p_6^2										X	X			
									X	X				
								X	X					
							X	X						
						X	X							
					X	X								
				X	X									
			X	X										
		X	X											p_6^1
	X	X										X	X	X
X	X											X	p_4	$X p_6^5$
X											p_6^1	$X p_6^3$	X	p_2

Figure 5: Attempt to Place p_6

9. $d(p_3, p_5) = \sqrt{5}$
10. $d(p_4, p_5) = \sqrt{290}$

We now try to place p_6 to create the fifth largest distance, $\sqrt{11^2 + 14^2} = \sqrt{317}$. We show this cannot be done. For $1 \le i \le 5$ we place p_6 in the place(s) it needs to be to get $d(p_i, p_6) = \sqrt{317}$. We label those places p_6^i.

Recall that we cannot use a distance twice.

1. For both p_6^1's, $d(p_6^1, p_4) = \sqrt{5} = d(p_3, p_5)$.
2. For one of the p_6^2, $d(p_6^2, p_5) = d(p_1, p_3) = 1$. For the other one $d(p_6^2, p_3) = d(p_1, p_5) = 2$.
3. p_6^3 is on an X spot.
4. There is no p_6^4 since it would be off the board.
5. p_6^5 is on an X spot.

▮

3 For all $n \geq 16$, n is NOT Placeable

Theorem 3.1

1. *Let $a, b, c, d \in \mathbb{N}$ with $(a, b) \neq (c, d)$ such that $4 \leq a, c$ and $1 \leq b, d \leq 3$. Then $a^2 + b^2 \neq c^2 + d^2$, so $5(a^2 + b^2) \neq 5(c^2 + d^2)$.*

2. *If $n \geq 29$ then $f(n) \geq n$.*

3. *If $n \geq 16$ then $f(n) \geq n$.*

4. *For all $n \geq 16$ the number of distances between spaces on the $n \times n$ chess board is $\leq \binom{n}{2} - 1$.*

5. *For all $n \geq 16$, n is not placeable.*

Proof:
1) There are three cases.
Case 1: $a = c$. Then we have $b \neq d$. We can assume $b^2 < d^2$. Hence we get

$$a^2 + b^2 < c^2 + d^2.$$

Case 2: $a < c$ so $c - a > 0$. Since $4 \leq a, c$ and $a < c$ we have $a + c \geq 9$. Using all of this we get:

$$c^2 - a^2 = (c + a)(c - a) \geq a + c \geq 9$$

so

$$a^2 + 9 \leq c^2.$$

Now we look at $a^2 + b^2$. Since $b \leq 3$, $b^2 \leq 9$. Hence

$$a^2 + b^2 \leq a^2 + 9 \leq c^2.$$

Since $d \geq 1$, $c^2 < c^2 + d^2$. Combining this with the above equation we get

$$a^2 + b^2 \leq a^2 + 9 \leq c^2 < c^2 + d^2.$$

Hence $a^2 + b^2 \neq c^2 + d^2$.
Case 3: $c < a$. Similar to Case 2.

2) For each (a, b) with $a > b > 0$ we can write $5(a^2 + b^2)$ as a sum of two squares in two different ways:

$$5(a^2 + b^2) = (|a - 2b|)^2 + (2a + b)^2 = (a + 2b)^2 + (2a - b)^2.$$

Since $a > b > 0$, we see that $2a+b$ is larger than the other numbers, so these representations are distinct.

For $n \geq 29$ we will use this to get many elements of $\{1, \ldots, n\}$ that can be written as the sum of two squares in two different ways. We need to split up cases for n even and n odd. Note that we will only need $2a + b \leq n$ since that is the largest number.

Case 0: n **even:** Let $4 \leq a \leq \frac{n-4}{2}$ and $1 \leq b \leq 3$. There are $\frac{n-10}{2} \times 3 = \frac{3n}{2} - 15$ pairs (a, b). When $n \geq 30$, this is $\geq n$.

We now show that $2a + b \leq n$.

$$2a + b \leq 2\left(\frac{n-4}{2}\right) + 3 < n$$

Case 1: n **odd:** Let $4 \leq a \leq \frac{n-3}{2}$ and $1 \leq b \leq 3$. We have $\frac{n-9}{2} \times 3 = \frac{3n}{2} - \frac{27}{2}$ pairs. When $n \geq 27$, this is $\geq n$.

We now show that $2a + b \leq n$.

$$2a + b \leq 2\left(\frac{n-3}{2}\right) + 3 = n.$$

3) Part 2 we only need to prove the theorem for $n = 16, \ldots, 28$.

By the proof of Theorem 2.3 we know that $f(15) \geq 14$. We indicate what numbers to add to that list of 14.

16: Add 225 and 250:

$225 = 0^2 + 15^2 = 9^2 + 12^2$

$250 = 5^2 + 15^2 = 9^2 + 13^2$

So $f(16) \geq 14 + 2 = 16$.

17,18: Add 260 and 265:

$260 = 2^2 + 16^2 = 8^2 + 14^2$

$265 = 11^2 + 12^2 = 3^2 + 16^2$

So $f(17) \geq 16 + 2 = 18$.

Hence $f(18) \geq 18$.

19,20,21,22: Add 365, 370, 377, 410:

$365 = 13^2 + 14^2 = 2^2 + 19^2$

$370 = 3^2 + 19^2 = 9^2 + 17^2$

$377 = 11^2 + 16^2 = 4^2 + 19^2$

$410 = 11^2 + 17^2 = 7^2 + 19^2$

So $f(19) \geq 18 + 4 = 22$.

Hence $f(20), f(21), f(22)$ are all ≥ 22.

23,24,25,26,27,28,29,30: Add 530, 533, 545, 565, 578, 610, 629, 650.

$530 = 13^2 + 19^2 = 1^2 + 23^2$
$533 = 2^2 + 23^2 = 7^2 + 22^2$
$545 = 16^2 + 17^2 = 4^2 + 23^2$
$565 = 6^2 + 23^2 = 9^2 + 22^2$
$578 = 17^2 + 17^2 = 7^2 + 23^2$
$610 = 13^2 + 21^2 = 9^2 + 23^2$
$629 = 10^2 + 23^2 = 2^2 + 25^2$
$650 = 11^2 + 23^2 = 17^2 + 19^2$
So $f(23) \geq 22 + 8 = 30$.
Hence $f(23), f(24), f(25), f(26), f(27), f(28), f(29), f(30)$ are all ≥ 30.

4) By Lemma 2.2 the number of distances is $\binom{n}{2} + n - 1 - f(n)$. By Part 3, $f(n) \geq n$.
Hence the number of distances is $\leq \binom{n}{2} - 1$.

5) Let $n \geq 16$. If n is placeable then there is a way to place n pennies on the $n \times n$ chessboard so that all $\binom{n}{2}$ distances occur. By Part 4 there are $\leq \binom{n}{2} - 1$ distances. Hence this cannot happen. ∎

4 Similar Problems

Erdős and Guy [1] posed the following question: Given n, what is the max k such that (k, n) is placeable? They showed that if (k, n) is placeable and n is large then, for all $\epsilon > 0$,

$$\Omega(n^{2/3-\epsilon}) \leq k \leq O\left(\frac{n}{(\log n)^{1/4}}\right).$$

The *Erdős Distance Problem* is the following: Given n points in the plane what is the minimum number of distinct distances. This is denoted by $g(n)$. Erdős showed that

$$\Omega(\sqrt{n}) \leq g(n) \leq O\left(\frac{n}{\sqrt{\log n}}\right).$$

Some sources say Erdős conjectured $(\forall c < 1)[g(n) = \Omega(n^c)]$. Some sources say that Erdős conjectured $g(n) \geq \Omega(\frac{n}{\sqrt{\log n}})$. There was steady progress on better lower bounds (see the Wikipedia page on *Erdős Distinct Distance Problem*) with the best result being by Guth and Katz [2], who showed $g(n) \geq \Omega(\frac{n}{\log n})$. This solves the first conjecture but not the second.

5 Open Problems

1. The placements for 3,4,5,6,7 are ad-hoc. We would like a theorem from which these (or some of these) placements are corollaries.

2. The proofs that, for $8 \leq n \leq 14$, n is not placeable was done by a computer. While we are confident that these proofs are valid, we would like to see a human-readable proof. Perhaps like the proof that 15 is not placeable. We suspect that (a) for $n = 14$ this is quite possible, though it may be a bit longer than we like, and (b) for $n = 8$ it will need new ideas.

3. What happens if you ask these problems in higher dimensions?

4. What happens with other metrics?

6 Acknowledgments

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References

[1] P. Erdős and R. Guy. Distinct distances between lattice points. *Elemente Der Mathematik*, 25:121–123, 1970.
https://users.renyi.hu/~p_erdos/1970-03.pdf.

[2] L. Guth and N. H. Katz. On the Erdős distinct distances problem in the plane. *Annals of Mathematics*, 181:155–190, 2015.

A Placements for 3,4,5,6,7

p_1		
	p_2	p_3

p_1			
p_2			p_3
			p_4

Figure 6: 3 and 4 are placeable

p_1				
p_2			p_3	
			p_4	
				p_5

			p_4		p_5
	p_3				
					p_6
p_2					
p_1					

Figure 7: 5 and 6 are placeable

		p_4				p_7
					p_6	
p_2	p_3					
p_1			p_5			

Figure 8: 7 is placeable