When Can One Load a Set of Dice so that the Sum is Uniformly Distributed?

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Introduction If you toss two fair six-sided dice, you will get a number between 2 and 12. Although each die is fair, the sum is not: the probability of getting a 2 is $\frac{1}{36}$, while the probability of a 7 is $\frac{1}{6}$. Thus the question arises: Can *unfair* (or loaded) dice lead to a fair sum? The answer is no, as was shown by Honsberger [**4**], using elementary methods. A proof using generating functions is in Hofri's book [**3**].

In [**1**], Chen, Rao, and Shreve raised the more general question of what happens with n m-sided dice; they showed that the answer is *still* no. In this note we generalize their result by considering m dice $D_1, \ldots, D_m$, where D_i is n_i-sided. We find that there are cases where one gets a fair sum, and we characterize exactly when this happens. Our techniques also lead to a different proof of the theorem of Chen, Rao, and Shreve.

A die is *fair* if all numbers appear with equal probability. A tuple of dice is *fair* if all sums appear with equal probability. For convenience, we will number an n-sided die $0, 1, \ldots, n-1$, and use the following definitions:

DEFINITIONS. *Let $m, n_1, \ldots, n_m \geq 2$, and let $N = \sum_{j=1}^{m} n_j$.*

- *An n-sided die is an ordered n-tuple of numbers $(p_0, \ldots, p_{n-1})$ such that (i) for all i, $0 \leq p_i \leq 1$; and (ii) $\sum_{i=0}^{n-1} p_i = 1$. (We think of p_i as the probability of rolling an i.)*
- *For $1 \leq j \leq m$ let D_j be an n_j-sided die. The m-tuple $(D_1, \ldots, D_m)$ is* fair *if, for all i, $0 \leq i \leq \sum_{j=1}^{m}(n_j - 1) = N - m$, the probability of rolling a sum of i is $\frac{1}{N-m+1}$.*
- *The ordered m-tuple $(n_1, \ldots, n_m)$ is* fair *if there exists $(D_1, \ldots, D_m)$ such that*

(i) each D_j is an n_j-sided die; and (ii) $(D_1, \ldots, D_m)$ is fair. In this case, we say that $(n_1, \ldots, n_m)$ is fair via *$(D_1, \ldots, D_m)$.*

Example. The ordered pair $(2, 3)$ is fair since $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2}))$ is fair. Every sum has probability $1/4$ of being rolled.

In the preceding example, every sum can be rolled in one and only one way. We will prove that, for a pair of dice to be fair, this condition *must* hold.

DEFINITIONS. *A die $D = (p_0, \ldots, p_{n-1})$ is* symmetric *if, for all i, $p_i = p_{n-1-i}$. D is* nice *if if it is symmetric and, for all i, either $p_i = 0$ or $p_i = p_0$.*

Note that if a die is nice then $p_0 \neq 0$—otherwise, $p_i = 0$ for all i.

MAIN THEOREM: *If $(D_1, \ldots, D_m)$ is fair then each D_i is nice.*

MAIN COROLLARY: *$(D_1, \ldots, D_m)$ is fair if and only if each D_i is nice and every sum can be rolled in exactly one way.*

ANOTHER MAIN COROLLARY: *There is a decision procedure that will, given $(n_1, \ldots, n_m)$, decide whether the tuple is fair.*

Proving the main theorem In this section we prove the main theorem. Our only tools are generating functions and some rudiments of complex algebra.

If $D = (p_0, \ldots, p_{n-1})$ is a die then the polynomial $F_D(z) = \sum_{i=0}^{n-1} p_i z^i$ is the *generating function* for D. The following key observation links tuples of dice and products of generating functions:

> If $(D_1, \ldots, D_m)$ is an m-tuple of dice, then the coefficient of z^i in $\prod_{i=1}^{m} F_{D_i}(z)$ is the probability of obtaining a sum of i.

The *kth roots of unity* are the complex solutions of the equation $z^k - 1 = 0$. It is easy to see that all of these roots lie on the complex unit circle. If k is even, then 1 and -1 are the only real roots of unity; if k is odd, then 1 is the only real root of unity. All the roots of unity have multiplicity 1.

LEMMA 1. *If $(n_1, \ldots, n_m)$ is fair via $(D_1, \ldots, D_m)$ and $N = \sum_{j=1}^{m} n_j$, then the roots of $\prod_{j=1}^{m} F_{D_j}(z)$ are exactly the $(N - m + 1)$th roots of unity except 1. Each root has multiplicity one.*

Proof. Since $(D_1, \ldots, D_m)$ is fair the probability of the sum being i is $\frac{1}{N-m+1}$. Hence

$$\prod_{j=1}^{m} F_{D_j}(z) = \sum_{i=0}^{N-m} \frac{z^i}{N-m+1} = \frac{(z^{N-m+1}-1)}{(N-m+1)(z-1)},$$

and it follows that $z^{N-m+1} - 1 = (N-m+1)(z-1)\prod_{j=1}^{m} F_{D_j}(z)$. ∎

We use Lemma 1 to restrict the kind of dice that can be used in an m-tuple of fair dice.

Lemma 2. *If $(D_1, \ldots, D_m)$ is fair, then each D_j is symmetric.*

Proof. Let $D_j = (p_0, \ldots, p_{n-1})$ and let $r_1, \ldots, r_{n-1}$ be the roots of $F_{D_j}(z)$. Since $F_{D_j}(z)$ has real coefficients and has roots on the unit circle

$$\{r_1, \ldots, r_{n-1}\} = \{\overline{r_1}, \ldots, \overline{r_{n-1}}\} = \{\frac{1}{r_1}, \ldots, \frac{1}{r_{n-1}}\}$$

where $\overline{r}_i$ denotes the complex conjugate of r_i. Hence the roots of $F_{D_j}(z)$ are the roots of $F_{D_j}(\frac{1}{z})$ which are also the roots of $z^{n-1}F_{D_j}(\frac{1}{z}) = \sum_{i=0}^{n-1} p_{n-1-i}z^i$. Since $\sum_{i=0}^{n-1} p_i z^i$ and $\sum_{i=0}^{n-1} p_{n-i}z^i$ have the same roots, the same degree, and $\sum_{i=0}^{n-1} p_i = 1$, these polynomials are identical. Hence $p_i = p_{n-1-i}$. ∎

We will prove our main theorem by induction on the number of dice. To this end, we need a way to combine two dice into one:

Definition. *$[D_1, D_2]$ is the die obtained by rolling dice D_1 and D_2 and considering their sum.*

The following lemma is key to the proof of the theorem.

Lemma 3. *If D_1 and D_2 are symmetric and $[D_1, D_2]$ is nice, then D_1 and D_2 are nice.*

Proof. Let $D_1 = (p_0, \ldots, p_{n_1-1})$ and $D_2 = (q_0, \ldots, q_{n_2-1})$. We assume $n_1 \leq n_2$. Since D_1 and D_2 are symmetric, $p_0 \neq 0$ and $q_0 \neq 0$. Let $[D_1, D_2] = (r_0, \ldots, r_{n_1+n_2-2})$.

We first prove that, for all i with $1 \leq i \leq n_1 - 1$, either $p_i = 0$ or $q_i = 0$. Since $[D_1, D_2]$ is nice $r_{n_2-1} = r_0$ or $r_{n_2-1} = 0$. Hence

$$p_0 q_{n_2-1} + p_1 q_{n_2-2} + \cdots + p_{n_1-1} q_{n_2-n_1} = a p_0 q_0,$$

where $a \in \{0, 1\}$.

Since $q_i = q_{n_2-1-i}$ for all i, we have $p_0q_0 + p_1q_1 + \cdots + p_{n_1-1}q_{n_1-1} = ap_0q_0$, so

$$p_1q_1 + \cdots + p_{n_1-1}q_{n_1-1} = (a-1)p_0q_0 \leq 0.$$

Since all the p_i and q_i are nonnegative, we have, for all i with $1 \le i \le n_1 - 1$, either $p_i = 0$ or $q_i = 0$.

We now prove that, for all i with $0 \le i \le n_1 - 1$, the following two conditions hold:

(i) either $p_i = 0$ or $p_i = p_0$; (ii) either $q_i = 0$ or $q_i = q_0$.

We prove this by induction on i. For $i = 0$ this is trivial. Assume it holds for all $i' < i$. Since $[D_1, D_2]$ is nice, either $r_i = r_0 = p_0 q_0$ or $r_i = 0$ for all i, so

$$p_0 q_i + p_1 q_{i-1} + \cdots + p_{i-1} q_1 + p_i q_0 = a p_0 q_0,$$

where $a \in \{0, 1\}$. By the induction hypothesis, each term $p_1 q_{i-1}, \ldots, p_{i-1} q_1$ is either 0 or $p_0 q_0$. Hence there exists $b \le a$ such that $p_0 q_i + b p_0 q_0 + p_i q_0 = a p_0 q_0$, so

$$(*) \qquad p_0 q_i + p_i q_0 = (a - b) p_0 q_0.$$

Note that $a - b \in \{0, 1\}$.

Now if $a - b = 1$, then $p_0 q_i + p_i q_0 = p_0 q_0$. If $p_i = 0$ (resp. $q_i = 0$) then $q_i = q_0$ (resp. $p_i = p_0$). By $(*)$, either $p_i = 0$ or $q_i = 0$.

If, alternatively, $a - b = 0$, then $p_0 q_i + p_i q_0 = 0$. Since $p_0 \ne 0$ and $q_0 \ne 0$, we have $p_i = q_i = 0$.

It remains to prove that, for all i with $n_1 \le i \le n_2 - 1$, either $q_i = 0$ or $q_i = q_0$. This is done by another induction on i, similar to the one just given. ∎

THEOREM 4. *If $(D_1, \ldots, D_m)$ is fair, then each D_j is nice.*

Proof. We prove this by induction on m. The $m = 1$ case is obvious. Assume the result holds for $m - 1$, and let $(D_1, \ldots, D_m)$ be fair. Then a simple calculation shows that $([D_1, D_2], D_3, D_4, \ldots, D_m)$ is fair. By the inductive hypothesis, each of the dice $[D_1, D_2], D_3, D_4, \ldots, D_m$ is nice. By Lemmas 2 and 3, D_1 and D_2 are nice. ∎

COROLLARY 5. *The tuple $(D_1, \ldots, D_m)$ is fair if and only if each D_i is nice and every sum can be rolled in exactly one way.*

Proof. Assume that $(D_1, \ldots, D_m)$ is fair, and that, for all j, D_j has n_j sides—we write $D_j = (p_{j0}, p_{j1}, p_{j2}, \ldots, p_{j(n_j-1)})$. Let $\mathrm{Prob}(a)$ denote the probability of rolling an a. Assume, by way of contradiction, that there exist a and distinct $(b_1, \ldots, b_m)$

and $(c_1, \ldots, c_m)$ such that

$$\text{(i)} \quad a = \sum_{j=1}^{m} b_j = \sum_{j=1}^{m} c_j; \qquad \text{(ii)} \quad \prod_{j=1}^{m} p_{j,b_j} \neq 0; \qquad \text{(iii)} \quad \prod_{j=1}^{m} p_{j,c_j} \neq 0.$$

Since $(b_1, \ldots, b_m)$ and $(c_1, \ldots, c_m)$ are two distinct ways of rolling an a, we have $\prod_{j=1}^{m} p_{j,b_j} + \prod_{j=1}^{m} p_{j,c_j} \leq \text{Prob}(a)$.

Since $(D_1, \ldots, D_m)$ is fair, each die is nice, so $\prod_{j=1}^{m} p_{j,b_j} = \prod_{j=1}^{m} p_{j,c_j} = \prod_{j=1}^{m} p_{j,0}$. Hence

$$2\text{Prob}(0) = \prod_{j=1}^{m} p_{j,0} + \prod_{j=1}^{m} p_{j,0} = \prod_{j=1}^{m} p_{j,b_j} + \prod_{j=1}^{m} p_{j,c_j} \leq \text{Prob}(a) = \text{Prob}(0).$$

This implies that $\text{Prob}(0) = 0$, which contradicts $(D_1, \ldots, D_m)$ being fair.

To prove the converse, assume that each D_i is nice and every sum can be rolled in exactly one way. Let a be rolled by $(b_1, \ldots, b_m)$ where, for all i, $p_{i,b_i} \neq 0$. Then the probability of rolling an a is $\prod_{j=1}^{m} p_{j,b_j} = \prod_{j=1}^{m} p_{j,0}$. Since this quantity is independent of a, $(D_1, \ldots, D_m)$ is fair. ∎

COROLLARY 6. *One can determine whether any given tuple $(n_1, \ldots, n_m)$ is fair.*

Proof. Given $(n_1, \ldots, n_m)$, we need only consider dice $(D_1, \ldots, D_m)$ with each D_i nice. There are only a finite number of possibilities; each one can be checked for fairness. ∎

The number of fair n-sided dice $(p_0, \ldots, p_{n-1})$ is the number of ways to assign values to $p_0, \ldots, p_{n-1}$ such that (1) for all i, $p_i = p_{n-1-i}$, (2) $p_0 \neq 0$, (3) for all i either $p_i = p_0$ or $p_i = 0$, and (4) $\sum_{i=0}^{n-1} = 1$. This is $\sum_{i=0}^{\lceil n/2 \rceil - 1} \binom{\lceil n/2 \rceil - 1}{i} = 2^{\lceil n/2 \rceil - 1}$. Thus the number of possibilities that must be considered is bounded by $\prod_{i=1}^{m} 2^{\lceil n_i/2 \rceil - 1}$.

Curious facts Next we explore some curious facts that follow from our work.

COROLLARY 7. *If a set of dice is fair, then at most one of them has an even number of sides.*

Proof. Assume, by way of contradiction, that $(D_1, \ldots, D_m)$ is fair and that n_i and n_j, $i \neq j$, are both even; then $n_i - 1$ and $n_j - 1$ are odd. Then the polynomials $F_{D_i}(z)$ and $F_{D_j}(z)$ have odd degree; since they also have real coefficients, both $F_{D_i}(z)$ and $F_{D_j}(z)$ must have real roots. Therefore $\prod_{j=1}^{m} F_{D_j}(z)$ either has at least two distinct real roots or one real root of multiplicity at least 2. The first possibility contradicts Lemma 1—there is at most one real $(N-m+1)$th root of unity other than 1, where $N=\sum_{j=1}^{m} n_j$. The second possibility also contradicts Lemma 1, since all roots of $\prod_{j=1}^{m} F_{D_j}(z)$ have multiplicity 1. ∎

The next corollary is the main theorem from [**1**]. We give an alternative proof.

COROLLARY 8. *If a set of dice is fair, then no two have the same number of sides.*

Proof. Suppose for the sake of contradiction that $n=n_i=n_j$, with $i\neq j$, and that $(D_1,\ldots,D_m)$ is fair. Let $D_k=(p_{k0},p_{k1},p_{k2},\ldots,p_{k(n_k-1)})$. Let $\mathrm{Prob}(n-1)$ be the probability of rolling an $n-1$. By Lemma 2, each D_k is symmetric, so $p_{i,n-1}=p_{i,0}$ and $p_{j,n-1}=p_{j,0}$. Hence

$$2\,\mathrm{Prob}(0)=2\prod_{k=1}^{m}p_{k,0}=\Big(p_{i,n-1}\,p_{j,0}\prod_{\substack{1\le k\le m\\ k\neq i,j}}p_{k,0}\Big)+\Big(p_{i,0}\,p_{j,n-1}\prod_{\substack{1\le k\le m\\ k\neq i,j}}p_{k,0}\Big)$$
$$\le \mathrm{Prob}(n-1)=\mathrm{Prob}(0).$$

Hence $\mathrm{Prob}(0)=0$, which contradicts $(D_1,\ldots,D_m)$ being fair. ∎

The next corollary mentions the Euler ϕ-function: for a positive integer n, $\phi(n)$ is the number of positive integers less than n that are relatively prime to n. The proof involves *cyclotomic polynomials*. For a positive integer n, the nth cyclotomic polynomial $\Phi_n(z)$ is a complex polynomial of degree $\phi(n)$; the roots of $\Phi_n(z)$ are the *primitive* nth roots of unity—those for which no lower power than n gives 1. (Curious readers may find more information on cyclotomic polynomials in any abstract algebra textbook; see, e.g., [**2**].)

COROLLARY 9. *If* $(n_1,\ldots,n_m)$ *is fair and* $N=\sum_{j=1}^{m} n_j$, *then* $\phi(N-m+1)\le \max_j(n_j-1)$. *Hence if* $N-m+1$ *is prime, then* $(n_1,\ldots,n_m)$ *is not fair.*

Proof. Assume $(n_1,\ldots,n_m)$ is fair via $(D_1,\ldots,D_m)$. By Theorem 4, each $F_{D_j}(z)$ has rational coefficients. By Lemma 1, a root of one of the $F_{D_j}(z)$ is a primitive $(N-m+1)$th root of unity. Therefore, $\Phi_{N-m+1}(z)$ divides some $F_{D_j}(z)$, where $\Phi_{N-m+1}(z)$ is the $(N-m+1)$th cyclotomic polynomial. Hence, for some j, $\phi(N-m+1)\le n_j-1$. ∎

Examples The following examples illustrate our results concretely.

1. By Corollary 7, no tuple of the form $(2, 2i)$ is fair.
2. All tuples of the form $(2, 2i{-}1)$, $i \geq 2$, are fair: use dice $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{i}, 0, \frac{1}{i}, 0, \ldots, \frac{1}{i}, 0, \frac{1}{i})$. (This produces $(2i)$-sided dice.)
3. All tuples $(i, i+1)$, $i \geq 2$, are fair: use dice $(\frac{1}{i}, \frac{1}{i}, \ldots, \frac{1}{i})$ and $(\frac{1}{2}, \overbrace{0, 0, \ldots, 0}^{i-1}, \frac{1}{2})$. (This produces $(2i)$-sided dice.)

The preceding examples show that, for $i \geq 3$, fair $(2i)$-sided dice can be produced in at least two different ways. For example, a six-sided die can be produced from $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3})$, and also from $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $(\frac{1}{2}, 0, 0, \frac{1}{2})$.

4. All tuples $(3, 3i{-}2)$, $i \geq 2$, are fair: use dice $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $(\frac{1}{i}, 0, 0, \frac{1}{i}, 0, 0, \ldots, \frac{1}{i}, 0, 0, \frac{1}{i})$. (This produces $(3i)$-sided dice.)
5. All tuples $(3, 4i{-}2)$, $i \geq 1$, are fair: use dice $(\frac{1}{2}, 0, \frac{1}{2})$ and $(\frac{1}{2i}, \frac{1}{2i}, 0, 0, \frac{1}{2i}, \frac{1}{2i}, 0, 0, \ldots, \frac{1}{2i}, \frac{1}{2i}, 0, 0, \frac{1}{2i}, \frac{1}{2i})$. (This produces $(4i)$-sided dice.)

The last two examples cover all the fair 2-tuples $(3, i)$, since we have exhausted all combinations of nice 3-sided dice. Some 2-tuples $(3, i)$ are fair in two different ways. For example, $(3, 10)$ is produced from $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $(\frac{1}{4}, 0, 0, \frac{1}{4}, 0, 0, \frac{1}{4}, 0, 0, \frac{1}{4})$, and also from $(\frac{1}{2}, 0, \frac{1}{2})$ and $(\frac{1}{6}, \frac{1}{6}, 0, 0, \frac{1}{6}, \frac{1}{6}, 0, 0, \frac{1}{6}, \frac{1}{6})$.

Finally, observe that one can construct fair dice from arbitrarily long tuples. All tuples of the form $(2^0 + 1, 2^1 + 1, 2^2 + 1, \ldots, 2^{m-1} + 1)$, $m \geq 2$, are fair: use dice $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0, \frac{1}{2})$, $(\frac{1}{2}, 0, 0, 0, \frac{1}{2})$, ..., $(\frac{1}{2}, \overbrace{0, 0, \ldots, 0}^{2^{m-1}-1}, \frac{1}{2})$. (This produces (2^m)-side dice.)

Almost uniform sums When giving a talk on this topic we were asked whether we can get "close to" a uniform sum using *real* dice. In this section, therefore, we assume dice are numbered from 1 to n.

There are several ways to measure how close a distribution is to uniform. We wrote a Matlab program to find, for given n, vectors $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ such that (i) $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$; (ii) for all i, $0 \leq p_i, q_i \leq 1$; and (iii) if we interpret the vectors as dice, then

$$\sum_{m=2}^{2n} \left(\text{Prob(sum is } m) - \frac{1}{2n-1} \right)^2$$

is minimized. For all n, Matlab produced symmetric dice that were identical to each other. However, Matlab *does not* guarantee that the results are the true optimum, so the question of whether or not optimal dice must be identical and symmetric is interesting and open, even in the cases where we obtained numerical results. Other measures of being "close to uniform" might also be considered.

We provide the statistics, in the $n = 6$ case, first for two (ordinary) fair dice and then for the dice we obtained from the program.

If each die is fair then the following happens.

$$\begin{aligned} \text{prob(the die is 1)} &= 0.166667 \\ \text{prob(the die is 2)} &= 0.166667 \\ \text{prob(the die is 3)} &= 0.166667 \\ \text{prob(the die is 4)} &= 0.166667 \\ \text{prob(the die is 5)} &= 0.166667 \\ \text{prob(the die is 6)} &= 0.166667 \end{aligned}$$

$$\begin{aligned} \text{prob(the sum is 2)} &= 0.027778 \\ \text{prob(the sum is 3)} &= 0.055556 \\ \text{prob(the sum is 4)} &= 0.083333 \\ \text{prob(the sum is 5)} &= 0.111111 \\ \text{prob(the sum is 6)} &= 0.138889 \\ \text{prob(the sum is 7)} &= 0.166667 \\ \text{prob(the sum is 8)} &= 0.138889 \\ \text{prob(the sum is 9)} &= 0.111111 \\ \text{prob(the sum is 10)} &= 0.083333 \\ \text{prob(the sum is 11)} &= 0.055556 \\ \text{prob(the sum is 12)} &= 0.027778 \end{aligned}$$

The two dice obtained by the Matlab program were unfair but identical, and had the following properties.

$$\begin{aligned} \text{prob(the die is 1)} &= 0.243883 \\ \text{prob(the die is 2)} &= 0.137480 \\ \text{prob(the die is 3)} &= 0.118637 \\ \text{prob(the die is 4)} &= 0.118637 \\ \text{prob(the die is 5)} &= 0.137480 \\ \text{prob(the die is 6)} &= 0.243883 \end{aligned}$$

$$\begin{aligned} \text{prob(the sum is 2)} &= 0.059479 \\ \text{prob(the sum is 3)} &= 0.067058 \\ \text{prob(the sum is 4)} &= 0.076768 \\ \text{prob(the sum is 5)} &= 0.090488 \\ \text{prob(the sum is 6)} &= 0.113753 \\ \text{prob(the sum is 7)} &= 0.184909 \\ \text{prob(the sum is 8)} &= 0.113753 \\ \text{prob(the sum is 9)} &= 0.090488 \\ \text{prob(the sum is 10)} &= 0.076768 \\ \text{prob(the sum is 11)} &= 0.067058 \\ \text{prob(the sum is 12)} &= 0.059479 \end{aligned}$$

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