Max and Min Limiters

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Abstract

If $A \subseteq \omega,\ n \geq 2$, and the function $\max(\{x_1, \ldots, x_n\} \cap A)$ is partial recursive, it is easily seen that A is recursive. In this paper, we weaken this hypothesis in various ways (and similarly for "min" in place of "max") and investigate what effect this has on the complexity of A. We discover a sharp contrast between retraceable and co-retraceable sets, and we characterize sets which are the union of a recursive set and a co-r.e., retraceable set. Most of our proofs are noneffective. Several open questions are raised.

1 Introduction

It is easy to see that, for $A \subseteq \omega$ and $n \geq 2$, if the (partial) function $\max(\{x_1, \dots, x_n\} \cap A)$ is partial recursive, then A is recursive. Suppose, on the other hand, that one can effectively eliminate a possibility for $\max(\{x_1, \dots, x_n\} \cap A)$, specifically, that one can, for any n distinct natural numbers $x_1, \dots, x_n$, calculate an i, $1 \leq i \leq n$, such that $x_i \neq \max(\{x_1, \dots, x_n\} \cap A)$. What does this tells us about A? In particular, must A be recursive? Suppose that one then weakens the assumption to the existence of a recursive function g such that, for all $x_1, \dots, x_n$, $W_{g(x_1,\dots,x_n)}$ is a proper subset of $\{x_1, \dots, x_n\}$ containing $\max(\{x_1, \dots, x_n\} \cap A)$. Do we now have more possibilities for the complexity of A? In this paper we consider questions of this sort, including analogous questions regarding "min" in place of "max." In many cases we find that the methods we use are at least as interesting as the statements of the theorems.

2 Background Material

Definition 2.1 Let $n, k \geq 1$, and let $f\colon \omega^k \to \omega$.

1. $f \in \mathrm{EN}(n)$ if there exist n partial recursive functions $\varphi_1, \dots, \varphi_n$ such that, for all $x_1, \dots, x_k$, there exists an i, $1 \leq i \leq n$, such that $\varphi_i(x_1, \dots, x_k)$ converges to $f(x_1, \dots, x_k)$. If $f \in \mathrm{EN}(n)$, we say that f is *n-enumerable*. Note that f is recursive iff f is 1-enumerable.

2. $f \in \mathrm{SEN}(n)$ if there exist n total recursive functions $g_1, \dots, g_n$ such that, for all $x_1, \dots, x_k$, there exists an i, $1 \leq i \leq n$, such that $g_i(x_1, \dots, x_k) = f(x_1, \dots, x_k)$. If $f \in \mathrm{SEN}(n)$, we say that f is *strongly n-enumerable*.

Definition 2.2 Let $A \subseteq \omega$, and $n \geq 1$.

1. $\mathrm{C}_n^A\colon \omega^n \to \{0,1\}^n$ is the function defined by

$$\mathrm{C}_n^A(x_1, \dots, x_n) = (\chi_A(x_1), \dots, \chi_A(x_n)),$$

   where χ_A is the characteristic function of A.

2. $\#_n^A\colon \omega^n \to \{0, \dots, n\}$ is the function defined by

$$\#_n^A(x_1, \dots, x_n) = |\{i : 1 \leq i \leq n \wedge x_i \in A\}|.$$

In his PhD thesis, Stanford University, 1987, Richard Beigel proved the following interesting theorem.

Theorem 2.3 (Nonspeedup Theorem [1, 2]) *If $(\exists n \geq 1)[\mathrm{C}_n^A \in \mathrm{EN}(n)]$, then A is recursive.*

Beigel conjectured that if $\#_n^A \in \mathrm{EN}(n)$ for some $n \geq 1$, then A is recursive. In 1987, Owings proved the following weak form of Beigel's conjecture.

Theorem 2.4 (Weak Cardinality Theorem [14]) *If $(\exists n \geq 1)[\#_n^A \in \mathrm{SEN}(n)]$, then A is recursive.*

Using entirely different methods, Martin Kummer proved Beigel's conjecture in 1990.

Theorem 2.5 (Cardinality Theorem [9]) *If* $(\exists n \geq 1)[\#_n^A \in \mathrm{EN}(n)]$, *then* A *is recursive.*

Kummer's proof of the Cardinality Theorem rested on the next two lemmas.

Definition 2.6 For $n \in \omega$, B_n is the full binary tree of height n.

Lemma 2.7 ([9]) *Let* $\mathcal{T}$ *be a subtree of* $2^{<\omega}$, *the full binary tree with* ω *levels, and let* $n \geq 1$. *If* B_{4^n-2} *can be embedded in* $\mathcal{T}$, *then there exist natural numbers* $x_1, \ldots, x_n$, *nodes* $\tau_1, \ldots, \tau_{n+1}$ *of* $\mathcal{T}$, *and* $b \in \{0, 1\}$ *such that, for* i, j *with* $1 \leq i \leq n$ *and* $1 \leq j \leq n+1$,

$$\tau_j(x_i) = \begin{cases} 1-b, & \text{if } 1 \leq i < j; \\ b, & \text{if } j \leq i \leq n. \end{cases}$$

Lemma 2.8 (R.E. Tree Lemma [9]) *let* $\mathcal{T}$ *be an r.e. subtree of* $2^{<\omega}$. *If, for some* $m \geq 1$, B_m *cannot be embedded in* $\mathcal{T}$, *then every infinite branch of* $\mathcal{T}$ *is recursive.*

In 1972, Jockusch and Soare proved the following important theorem.

Theorem 2.9 ([7]) *If a function* f *is recursive in every member of some nonempty* Π_1^0 *class of subsets of* ω, *then* f *is recursive.*

Definition 2.10 For $A \subseteq \omega$, A is *extensive* if every partial recursive function φ having a finite range can be extended to a total function f which is recursive in A.

(The Turing degrees of the extensive sets are the same as the Turing degrees of the consistent extensions of Peano arithmetic [4]. They are called *DNR*$_2$ [4] or *PA* [8].)

It is well-known, and easy to prove, that there exists a nonempty Π_1^0 class all of whose members are extensive; thus we have the following proposition.

Proposition 2.11 *If a function* f *is recursive in every extensive set, then* f *is recursive.*

As observed by Kummer and Stephan [11], it follows from Owings' proof of the Weak Cardinality Theorem that if $\#_n^A \in \mathrm{EN}(n)$ for some $n \geq 1$, then A is recursive in every extensive set. So there exist two completely different—and seemingly unrelated—proofs of the Cardinality Theorem.

Recently, Jockusch has extended Theorem 2.9 as follows:

Theorem 2.12 ([5]) *If* A *is r.e. in every member of some nonempty* Π_1^0 *class of subsets of* ω, *then* A *is r.e.*

Thus we get the following proposition.

Proposition 2.13 *If* A *is r.e. in every extensive set, then* A *is r.e.*

For completeness, we include Jockusch's proof of Theorem 2.12.

Proof of Theorem 2.12: Let $\mathcal{T}$ be a recursive tree with at least one infinite branch. Suppose that A is r.e. in every infinite branch of $\mathcal{T}$, but that A is not r.e.

We inductively define an infinite sequence $(\mathcal{T}_{-1}, \mathcal{T}_0, \mathcal{T}_2, \ldots)$ of infinite recursive trees so that

- for every e, $\mathcal{T}_e$ is a subtree of $\mathcal{T}_{e-1}$, and
- the intersection of all these trees has at least one infinite branch.

Let $\mathcal{T}_{-1} = \mathcal{T}$. Assume that $\mathcal{T}_{e-1}$ has been defined, and that $\mathcal{T}_{e-1}$ is an infinite recursive tree. For every $n \in \omega$, let $\mathcal{T}_{e,n} = \{\sigma \in \mathcal{T}_{e-1} : \{e\}^{\sigma}_{|\sigma|}(n) \uparrow\}$. Then let $B_e = \{n : \mathcal{T}_{e,n} \text{ is finite}\}$. For every n with $\mathcal{T}_{e,n}$ finite, there exists a natural number m such that $\{e\}^{\sigma}_{|\sigma|}(n) \downarrow$ for all $\sigma \in \mathcal{T}_{e-1}$ with $|\sigma| = m$, and such an m can be found effectively. Therefore, B_e is r.e.; moreover, $A \neq B_e$, since A is not r.e. To define $\mathcal{T}_e$, we have two cases to consider.

- $B_e \not\subseteq A$: Let $\mathcal{T}_e = \mathcal{T}_{e-1}$.
- $B_e \subseteq A$: Then $B_e \subset A$, so choose $n \in A - B_e$ and let $\mathcal{T}_e = \mathcal{T}_{e,n}$.

In either case, $\mathcal{T}_e$ is an infinite recursive subtree of $\mathcal{T}_{e-1}$; furthermore, $A \neq W_e^C$ for *any* infinite branch C of $\mathcal{T}_e$.

Let $\mathcal{T}_\omega = \bigcap\{\mathcal{T}_e : e \geq -1\}$. Then $\mathcal{T}_\omega$ is an infinite closed subtree of $\mathcal{T}$, and so has an infinite branch C. However, $A \neq W_e^C$ for *any* e, contradicting our hypothesis that A is r.e. in every infinite branch of $\mathcal{T}$. ∎

Definition 2.14 Let $n \geq 1$. An *n-ary selector function* is a function $f\colon \omega^n \to \omega$ such that $f(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$ for all $x_1, \ldots, x_n$.

Definition 2.15 For $A \subseteq \omega$, A is *semirecursive* if there is a recursive selector function $f\colon \omega^2 \to \omega$ such that, for all x_1, x_2, $\{x_1, x_2\} \cap A \neq \emptyset \Rightarrow f(x_1, x_2) \in A$.

Kummer and Stephan [12] have shown that if A is r.e., then A is semirecursive iff $\mathrm{C}_2^A \in \mathrm{SEN}(3)$. The proof we give here is adapted from a proof given in an unpublished technical paper [10].

Proposition 2.16 ([10, 12]) *Let A be r.e. Then A is semirecursive iff $\mathrm{C}_2^A \in \mathrm{SEN}(3)$.*

Proof:

$\Rightarrow$: Suppose that A is semirecursive via selector function f. We define (total) recursive functions g_1, g_2, g_3 so that, for all x, y,

$$\mathrm{C}_2^A(x, y) \in \{g_1(x, y), g_2(x, y), g_3(x, y)\}.$$

Let $x, y \in \omega$, and suppose that $f(x, y) = x$. (The case where $f(x, y) = y$ is similar.) Let $g_1(x, y) = (1, 1)$, $g_2(x, y) = (1, 0)$, and $g_3(x, y) = (0, 0)$.

$\Leftarrow$: Suppose that $\mathrm{C}_2^A \in \mathrm{SEN}(3)$, and note that this is equivalent to the existence of a recursive function $g\colon \omega \times \omega \to \{0, 1\} \times \{0, 1\}$ such that, for all x, y, either $(g(x, y))_0 = \chi_A(x)$ or $(g(x, y))_1 = \chi_A(y)$. Also, we can assume that $g(y, x) = ((g(x, y))_1, (g(x, y))_0)$.

Let

$$\begin{aligned} B_1 &= \{z : (\exists x \in A)[g(x,z) = (0,0)]\}, \\ B_2 &= \{z : (\exists x \in B_1)[g(x,z) = (1,0)]\}. \end{aligned}$$

Then B_1 and B_2 are r.e., and $B_1, B_2 \subseteq \overline{A}$.

If $B_1 \cup B_2 = \overline{A}$, then A is recursive and, therefore, semirecursive. So assume that $B_1 \cup B_2 \subset \overline{A}$, and choose $u \in \overline{A} - (B_1 \cup B_2)$. We define a recursive selector function f so that A is semirecursive via f.

Let $x, y \in \omega$. If $g(x,y) = (1,1)$, enumerate A until x or y appears and let $f(x,y)$ be whichever one appears first. If $g(x,y) = (0,1)$, let $f(x,y) = y$; if $g(x,y) = (1,0)$, let $f(x,y) = x$. If $g(x,y) = (0,0)$, use the value of $g(x,u)$ to define $f(x,y)$; there are four cases to consider.

- $g(x,u) = (1,1)$: Then $x \in A$, so let $f(x,y) = x$.
- $g(x,u) = (0,1)$: Then $x \notin A$, so let $f(x,y) = y$.
- $g(x,u) = (0,0)$: Then $x \notin A$, else $u \in B_1$, so let $f(x,y) = y$.
- $g(x,u) = (1,0)$: Since $g(x,y) = (0,0) = g(y,x)$, it must be the case that $y \notin A$, since

  $$y \in A \Rightarrow x \in B_1 \Rightarrow u \in B_2,$$

  in contradiction to the choice of u. Thus let $f(x,y) = x$.

∎

3 Preliminaries

Definition 3.1 Let $A \subseteq \omega$, and let $n \geq 1$. IFIRST_n^A, ILAST_n^A, IMIN_n^A, and IMAX_n^A are the n-ary functions defined as follows.

1.

$$\mathrm{IFIRST}_n^A(x_1, \ldots, x_n) = \begin{cases} \text{least } i \text{ such that } x_i \in A, & \text{if } (\exists i)[x_i \in A]; \\ 0, & \text{if } (\forall i)[x_i \notin A]. \end{cases}$$

2.

$$\mathrm{ILAST}_n^A(x_1, \ldots, x_n) = \begin{cases} \text{greatest } i \text{ such that } x_i \in A, & \text{if } (\exists i)[x_i \in A]; \\ 0, & \text{if } (\forall i)[x_i \notin A]. \end{cases}$$

3.

$$\begin{aligned} \mathrm{IMIN}_n^A(x_1, \ldots, x_n) &= \mathrm{IFIRST}_n^A(y_1, \ldots, y_n), \\ \mathrm{IMAX}_n^A(x_1, \ldots, x_n) &= \mathrm{ILAST}_n^A(y_1, \ldots, y_n), \end{aligned}$$

where $(y_1, \ldots, y_n)$ is the unique rearrangement of $(x_1, \ldots, x_n)$ satisfying $y_1 \leq y_2 \leq \cdots \leq y_n$. (For example, if $n = 4$ and $(x_1, x_2, x_3, x_4) = (3,4,1,3)$, then $(y_1, y_2, y_3, y_4) = (1,3,3,4)$.) Note that IMIN_n^A and IMAX_n^A are symmetric functions.

Definition 3.2 Let $A \subseteq \omega$, and let $n \geq 1$.

1. A has an *n-ary IMAX-limiter* if $\mathrm{IMAX}_n^A \in \mathrm{EN}(n)$.
2. A has a strong *n-ary IMAX-limiter* if $\mathrm{IMAX}_n^A \in \mathrm{SEN}(n)$.
3. Let g be a recursive n-ary function.
   (a) g is an *n-ary IMAX-limiter for A* if, for all $x_1, \ldots, x_n$, $\mathrm{IMAX}_n^A(x_1, \ldots, x_n) \in W_{g(x_1,\ldots,x_n)} \subset \{0, \ldots, n\}$.
   (b) g is a *strong n-ary IMAX-limiter for A* if, for all $x_1, \ldots, x_n$, $\mathrm{IMAX}_n^A(x_1, \ldots, x_n) \in D_{g(x_1,\ldots,x_n)} \subset \{0, \ldots, n\}$. (For $e \in \omega$, D_e is the e^{th} canonical finite set: $D_0 = \emptyset$; for $e > 0$, D_e consists of the natural numbers $y_1, \ldots, y_k$ such that $e = \Sigma_{i=1}^k 2^{y_i}$.)
4. *n-ary IMIN-limiter* and *strong n-ary IMIN-limiter* are defined analogously, with IMIN in place of IMAX.

Definition 3.3 Let $A \subseteq \omega$.

1. A is *retraceable* if there is a partial recursive function ψ such that
   - if a_1 is the least element of A, then $\psi(a_1) \downarrow = a_1$, and
   - for every $n \geq 1$, if a_n and a_{n+1} are the n^{th} smallest and $n + 1^{\text{st}}$ smallest elements of A, respectively, then $\psi(a_{n+1}) \downarrow = a_n$.

   (Note that every recursive set is retraceable.) If such a ψ exists, ψ is said to be a *retracing function for A*.
2. A is *strongly retraceable* if A has a retracing function that is (total) recursive.

In this paper we shall attempt to characterize, for a fixed natural number $n \geq 2$, those sets A for which one of the four n-ary functions defined above is either n-enumerable or strongly n-enumerable. In general, we obtain only partial results. For example, we show that $\mathrm{IMAX}_2^A \in \mathrm{SEN}(2)$ iff $\overline{A}$ is the union of a recursive set and a strongly retraceable set, but we obtain no characterization of sets A such that $\mathrm{IMAX}_2^A \in \mathrm{EN}(2)$.

One of our results tells us that a nonrecursive set cannot have both an IMAX-limiter and an IMIN-limiter.

The reader may wonder why we have not considered the functions $\mathrm{MAX}_n^A(x_1, \ldots, x_n) = \max(A \cap \{x_1, \ldots, x_n\})$ and $\mathrm{MIN}_n^A(x_1, \ldots, x_n) = \min(A \cap \{x_1, \ldots, x_n\})$. The main reason is that these functions are not defined when $\{x_1, \ldots, x_n\} \cap A = \emptyset$. What would it mean for a set A to have a MAX-limiter? We feel the following definition is the most natural.

Definition 3.4 Let $A \subseteq \omega$, and let $n \geq 2$.

1. A has an *n-ary MAX-limiter* if there is an n-ary selector function $f \in \mathrm{EN}(n-1)$ such that $f(x_1, \ldots, x_n) = \max(\{x_1, \ldots, x_n\} \cap A)$ whenever $\{x_1, \ldots, x_n\} \cap A \neq \emptyset$.
2. *strong n-ary MAX-limiter*, *n-ary MIN-limiter*, and *strong n-ary MIN-limiter* are defined analogously.

Most questions concerning MAX- and MIN-limiters can be reduced to questions about IMAX- and IMIN-limiters—or can be settled by similar techniques. Thus, in this paper, we shall not deal extensively with these types of limiters.

4 Results

Theorem 4.1 *If A has an n-ary IMIN-limiter for some $n \geq 1$, then A is co-r.e.*

Proof: First, we suppose that A has a strong n-ary IMIN-limiter for some $n \geq 1$. We proceed by induction on n. If $n = 1$, then IMIN_n^A—and therefore A—is recursive. If $n > 1$, choose a recursive n-ary function g so that, for all $x_1, \ldots, x_n$,

$$\text{IMIN}_n^A(x_1, \ldots, x_n) \in D_{g(x_1,\ldots,x_n)} \subset \{0, \ldots, n\}.$$

There are two cases.

- For every $x_1 \notin A$, there exist $x_2, \ldots, x_n$ such that $x_1 \leq x_2 \leq \cdots \leq x_n$ and $1 \notin D_{g(x_1,\ldots,x_n)}$: Then

  $$\overline{A} = \{x_1 : (\exists x_2, \ldots, x_n)[x_1 \leq x_2 \leq \cdots \leq x_n \wedge 1 \notin D_{g(x_1,\ldots,x_n)}]\,\}.$$

  In this case, A is clearly co-r.e.

- There exists $x_1 \notin A$ such that $1 \in D_{g(x_1,\ldots,x_n)}$ for all $x_2, \ldots, x_n$ with $x_1 \leq x_2 \leq \cdots \leq x_n$: Using the values of $A(0), \ldots, A(x_1)$, we will define a recursive $(n-1)$-ary function h so that, for all $y_1, \ldots, y_{n-1}$,

  $$\text{IMIN}_n^A(y_1, \ldots, y_{n-1}) \in D_{h(y_1,\ldots,y_{n-1})} \subset \{0, \ldots, n-1\}.$$

  Using the induction hypothesis, it then follows that A is co-r.e.

  Let $y_1, \ldots, y_{n-1} \in \omega$. We can assume that $y_1 \leq \cdots \leq y_{n-1}$. If $y_1 \leq x_1$, define $h(y_1, \ldots, y_{n-1})$ by

  $$D_{h(y_1,\ldots,y_{n-1})} = \begin{cases} \{1, 2, \ldots, n-1\}, & \text{if } y_1 \in A; \\ \{0, 2, \ldots, n-1\}, & \text{if } y_1 \notin A. \end{cases}$$

  (If $n = 2$, $D_{h(y_1,\ldots,y_{n-1})}$ is defined to be $\{1\}$ if $y_1 \in A$, and $\{0\}$ otherwise.) If $x_1 < y_1$, define $h(y_1, \ldots, y_{n-1})$ by

  $$D_{h(y_1,\ldots,y_{n-1})} = \{k : (k = 0 \wedge k \in D_{g(x_1,y_1,\ldots,y_{n-1})}) \vee (k > 0 \wedge k+1 \in D_{g(x_1,y_1,\ldots,y_{n-1})})\}.$$

If the n-ary IMIN-limiter for A is not strong, we must replace $D_{g(x_1,\ldots,x_n)}$ by $W_{g(x_1,\ldots,x_n)}$. However, if C is any extensive set, there exists a function h recursive in C such that $W_{g(x_1,\ldots,x_n)} \subseteq D_{h(x_1,\ldots,x_n)} \subset \{0, \ldots, n\}$. We then mimic the foregoing proof and conclude that $\overline{A}$ is r.e. in C. By Proposition 2.13, $\overline{A}$ is r.e. ∎

We have no idea how to prove Theorem 4.1 without appealing to Proposition 2.13. Perhaps it can be done, but the argument may be much more intricate.

The following refinement of Theorem 4.1 will prove useful.

Proposition 4.2 *A has a strong binary IMIN-limiter g such that $0 \in D_{g(x,y)}$ for all x, y iff A is retraceable and co-r.e.*

Proof:

$\Rightarrow$: Suppose that A has a strong binary IMIN-limiter g such that $0 \in D_{g(x,y)}$ for all x, y. We can assume that A is nonempty, and that $|D_{g(x,y)}| = 2$ for all x, y. By Theorem 1, we know that A is co-r.e. We define a (total) recursive function f that retraces A. (Note that this will actually show that A is *strongly* retraceable. This will not pose a problem in the proof of the converse, since it has been shown by Dekker and Myhill [3] that every co-r.e. set which is retraceable is strongly retraceable.) Let a be the smallest element of A.

Let $f(a) = a$ and, for $y \in \omega - \{a\}$, let $f(y)$ be the greatest number $x < y$ such that $D_{g(x,y)} = \{0, 1\}$, if such an x exists; otherwise, let $f(y) = y$. Note that, for every $y \in A$ with $y > a$, $f(y)$ is the greatest number $x < y$ such that $x \in A$; hence f retraces A.

$\Leftarrow$: Suppose that A is retraceable and co-r.e. Choose a (total) recursive retracing function f for A. (See [3] for a proof that such an f exists.) We define a strong binary IMIN-limiter g for A so that $0 \in D_{g(x,y)}$ for all x, y.

Let $x, y \in \omega$. We can assume that $x \leq y$. If $x = y$, let $D_{g(x,y)} = \{0, 1\}$. If $x < y$, simultaneously enumerate $\overline{A}$ and compute $f(y), f(f(y)), \ldots$. Define $D_{g(x,y)}$ according to which of the following three events occurs first:

- x enters $\overline{A}$: Set $D_{g(x,y)} = \{0, 2\}$.
- y enters $\overline{A}$: Set $D_{g(x,y)} = \{0, 1\}$.
- For some n, $x = f^{(n)}(y)$ (note that this will occur if $\{x, y\} \subseteq A$; moreover, if $x = f^{(n)}(y)$, then $y \in A \Rightarrow x \in A$): Set $D_{g(x,y)} = \{0, 1\}$.

∎

Corollary 4.3 *A has a binary MIN-limiter iff A is co-r.e. and retraceable.*

Proof:

$\Rightarrow$: Suppose that A has a binary MIN-limiter f. Then $f \in \mathrm{EN}(1)$ (hence f is recursive) and $f(x, y) = \min(A \cap \{x, y\})$ whenever $A \cap \{x, y\} \neq \emptyset$. We define a strong binary IMIN-limiter g for A so that $0 \in D_{g(x,y)}$ for all x, y, from which it follows (by Proposition 4.2) that A is co-r.e. and retraceable.

Let $x, y \in \omega$. We can assume that $x \leq y$. If $x = y$, let $D_{g(x,y)} = \{0, 1\}$. If $x < y$, let

$$D_{g(x,y)} = \begin{cases} \{0,1\}, & \text{if } f(x,y) = x; \\ \{0,2\}, & \text{if } f(x,y) = y. \end{cases}$$

$\Leftarrow$: Suppose that A is co-r.e. and retraceable. By Proposition 4.2, A has a strong binary IMIN-limiter g such that $0 \in D_{g(x,y)}$ for all x, y. We define a binary MIN-limiter f for A.

Let $x, y \in \omega$. We can assume that $x \neq y$. If $x < y$ (the case $y < x$ is similar), let

$$f(x,y) = \begin{cases} x, & \text{if } D_{g(x,y)} \subseteq \{0,1\}; \\ y, & \text{if } D_{g(x,y)} = \{0,2\}. \end{cases}$$

∎

Theorem 4.4 *A has a strong binary IMIN-limiter iff A is the union of a recursive set and a co-r.e., retraceable set.*

Proof:

$\Rightarrow$: Suppose that A has a strong binary IMIN-limiter g. We can assume that $|D_{g(x,y)}| = 2$ for all x, y. First, we show that $\mathrm{C}_2^A \in \mathrm{SEN}(3)$, i.e., we define a recursive binary function f so that, for all x, y, $\mathrm{C}_2^A(x,y) \in D_{f(x,y)} \subset \{(0,0),(0,1),(1,0),(1,1)\}$.

Let $x, y \in \omega$. If $x = y$, let $D_{f(x,y)} = \{(0,0),(1,1)\}$. If $x < y$, let

$$D_{f(x,y)} = \begin{cases} \{(1,0),(0,1),(1,1)\}, & \text{if } 0 \notin D_{g(x,y)}; \\ \{(0,0),(0,1)\}, & \text{if } 1 \notin D_{g(x,y)}; \\ \{(0,0),(1,0),(1,1)\}, & \text{if } 2 \notin D_{g(x,y)}. \end{cases}$$

If $x > y$, let $D_{f(x,y)} = \{(i,j) : (j,i) \in D_{f(y,x)}\}$.

By Theorem 4.1, A is co-r.e. Moreover, since $\mathrm{C}_2^A \in \mathrm{SEN}(3)$, it follows easily that $\mathrm{C}_2^{\overline{A}} \in \mathrm{SEN}(3)$. Thus, by Proposition 2.16, $\overline{A}$ is semirecursive. From this it follows that A is semirecursive. Choose a recursive binary selector function h so that A is semirecursive via h.

Define a recursive set B by

$$B = \{y : (\exists x < y)[0 \notin D_{g(x,y)} \ \wedge \ h(x,y) = y]\}.$$

Then $B \subseteq A$. Let $C = A - B$. Now C is co-r.e., since A is co-r.e. and B is recursive.

It remains to show that C is retraceable. We define a strong binary IMIN-limiter k for C so that $0 \in D_{k(x,y)}$ for all x, y, from which it follows (by Proposition 4.2) that C is retraceable.

Let $x, y \in \omega$. We can assume that $x \le y$. If $x = y$, let $D_{k(x,y)} = \{0,1\}$. If $x < y$, let

$$D_{k(x,y)} = \begin{cases} \{0,2\}, & \text{if } x \in B; \\ D_{g(x,y)}, & \text{if } 0 \in D_{g(x,y)} \wedge x \notin B; \\ \{0,1\}, & \text{otherwise.} \end{cases}$$

Note that if $x < y$ and the first two clauses in the definition of $k(x,y)$ fail, then $0 \notin D_{g(x.y)}$ and $x \notin B$. If $y \in B$, then $y \notin C$, since $C \subseteq \overline{B}$. If, on the other hand, $y \notin B$, then (by the definition of B and the fact that $0 \notin D_{g(x,y)}$) $h(x,y) = x$. In either case, $\{x,y\} \cap C \neq \emptyset \Rightarrow x \in C$. Thus k is as claimed.

$\Leftarrow$: Suppose that $A = B \cup C$, where B is recursive and C is co-r.e. and retraceable. We define a strong binary IMIN-limiter g for A.

Let $x, y \in \omega$. We can assume that $x \le y$. If $x = y$, let $D_{g(x,y)} = \{0,1\}$. If $x < y$, let $D_{g(x,y)} = \{1,2\}$ if $\{x,y\} \cap B \neq \emptyset$; otherwise (i.e., if $\{x,y\} \cap B = \emptyset$), define $D_{g(x,y)}$ as in the proof of Proposition 4.2 (using C and $\overline{C}$ in place of A and $\overline{A}$, respectively). ∎

We cannot characterize the class $\{A : A \text{ has a binary IMIN-limiter}\}$. One reason is that Proposition 2.16 fails if SEN is replaced by EN: It is well known (see [6]) that there exist r.e. sets that are not semirecursive; note, however, that if A is any r.e. set, then $\mathrm{C}_2^A \in \mathrm{EN}(3)$. Also, we have not characterized any of the classes $\{A : A \text{ has a strong } n\text{-ary IMIN-limiter}\}$, $n > 2$.

By Theorem 4.4, $\mathrm{IMIN}_2^A \in \mathrm{SEN}(2)$ iff A is the union of a recursive set and a co-r.e., retraceable set. Martin [13] has shown that $\mathrm{IFIRST}_2^A \in \mathrm{SEN}(2)$ iff A is recursive. The following theorem is an extension of her result.

Theorem 4.5 *If* $\mathrm{IFIRST}_2^A \in \mathrm{EN}(2)$*, then* A *is recursive.*

Proof: First, we consider the case where $\mathrm{IFIRST}_2^A \in \mathrm{SEN}(2)$ and give Martin's argument. By assumption, there exists a recursive binary function f such that, for all x_1, x_2, $\mathrm{IFIRST}_2^A(x_1, x_2) \in D_{f(x_1,x_2)} \subset \{0, 1, 2\}$. We can assume that $|D_{f(x_1,x_2)}| = 2$ for all (x_1, x_2). We define a recursive binary function g so that, for all x_1, x_2, $\#_2^A(x_1, x_2) \in D_{g(x_1,x_2)} \subset \{0, 1, 2\}$. By the Weak Cardinality Theorem, A is recursive.

Let $x_1, x_2 \in \omega$. If $x_1 = x_2$, let $D_{g(x_1,x_2)} = \{0, 2\}$. So assume that $x_1 \neq x_2$.

If $D_{f(x_1,x_2)} = \{1, 2\}$ or $D_{f(x_2,x_1)} = \{1, 2\}$, then $A \cap \{x_1, x_2\} \neq \emptyset$, so let $D_{g(x_1,x_2)} = \{1, 2\}$. If $D_{f(x_1,x_2)} = \{0, 2\}$ or $D_{f(x_2,x_1)} = \{0, 2\}$, then $\overline{A} \cap \{x_1, x_2\} \neq \emptyset$, so let $D_{g(x_1,x_2)} = \{0, 1\}$. If neither of these conditions is satisfied, then $D_{f(x_1,x_2)} = \{0, 1\} = D_{f(x_2,x_1)}$, hence $|A \cap \{x_1, x_2\}| \neq 1$, so let $D_{g(x_1,x_2)} = \{0, 2\}$.

Now suppose that $\mathrm{IFIRST}_2^A \in \mathrm{EN}(2)$, and let B be any extensive set. Then there exists a binary function f recursive in B such that, for all x_1, x_2, $\mathrm{IFIRST}_2^A(x_1, x_2) \in D_{f(x_1,x_2)} \subset \{0, 1, 2\}$. By a relativization of the above argument, A is recursive in B. Hence by Proposition 2.11, A is recursive. ∎

We now investigate which sets have IMAX-limiters.

Theorem 4.6 *If* A *is a retraceable set such that* $\mathrm{IMAX}_n^A \in \mathrm{EN}(n)$ *for some* $n \geq 1$*, then* A *is recursive.*

Proof: Suppose that A is a retraceable set such that $\mathrm{IMAX}_n^A \in \mathrm{EN}(n)$ for some $n \geq 1$. Choose a retracing function ψ for A. For $x_1, \ldots, x_n \in \omega$ with $x_1 \leq \cdots \leq x_n$, $\mathrm{C}_n^A(x_1, \ldots, x_n)$ can be readily calculated from $\mathrm{IMAX}_n^A(x_1, \ldots, x_n)$ and ψ. Hence $\mathrm{C}_n^A \in \mathrm{EN}(n)$. Thus A is recursive, by the Nonspeedup Theorem. ∎

In contrast to Theorem 4.6, we have the following theorem, from which it follows that if $\overline{A}$ is retraceable, then $\mathrm{IMAX}_n^A \in \mathrm{EN}(2)$.

Theorem 4.7 *Let* $n \geq 1$*. If* $\overline{A}$ *is the union of a recursive set and a retraceable set, then* $\mathrm{IMAX}_n^A \in \mathrm{EN}(2)$*.*

Proof: Suppose that $\overline{A} = B \cup C$, where B is recursive and C is retraceable. Choose a retracing function ψ for C. We define partial recursive functions φ_1, φ_2 so that, for all $x_1, \ldots, x_n \in \omega$, at least one of the functions φ_1, φ_2 converges to $\mathrm{IMAX}_n^A(x_1, \ldots, x_n)$ on input $(x_1, \ldots, x_n)$.

Let $x_1, \ldots, x_n \in \omega$. We can assume that $x_1 \leq \cdots \leq x_n$. Compute $m = |\{x_1, \ldots, x_n\} \cap \overline{B}|$. If $m = 0$, then $\{x_1, \ldots, x_n\} \subseteq B$ (hence $\mathrm{IMAX}_n^A(x_1, \ldots, x_n) = 0$, since $A \subseteq \overline{B}$), so let $\varphi_1(x_1, \ldots, x_n) = 0$ and let $\varphi_2(x_1, \ldots, x_n)$ diverge.

If $m > 0$, let $y_1, \ldots, y_m$ be such that $\{y_1, \ldots, y_m\} = \{x_1, \ldots, x_n\} \cap \overline{B}$ and $m > 1 \Rightarrow y_1 < \cdots < y_m$. (Note that $y_1, \ldots, y_m$ are the only candidates for membership in $\{x_1, \ldots, x_n\} \cap A$, since $A \subseteq \overline{B}$.) We define $\varphi_1(x_1, \ldots, x_n), \varphi_2(x_1, \ldots, x_n)$ so that if $y_m \in A$, then $\varphi_1(x_1, \ldots, x_n)$ will output $\mathrm{IMAX}_n^A(x_1, \ldots, x_n)$; otherwise, $\varphi_2(x_1, \ldots, x_n)$ will do so.

First, consider the possibility that $y_m \in A$. Let $r = \max(\{i : x_i = y_m\})$, and note that $y_m \in A \Rightarrow \mathrm{IMAX}_n^A(x_1, \ldots, x_n) = r$, so let $\varphi_1(x_1, \ldots, x_n) = r$.

Next, consider the possibility that $y_m \notin A$ (in which case $y_m \in C$, since $y_m \in \overline{B}$ and $A = \overline{B} \cap \overline{C}$). Search for $k \geq 1$ and $w_1, \ldots, w_k \in \omega$ such that

- $w_k = y_m$,
- $\psi(w_1) \downarrow= w_1$,
- for every i with $1 \leq i < k$, $\psi(w_{i+1}) \downarrow= w_i$, and
- $k > 1 \Rightarrow w_1 < \cdots < w_k$.

If such $k, w_1, \ldots, w_k$ are found (in which case $y_m \in \overline{A} \Rightarrow y_m \in C \Rightarrow \{w_1, \ldots, w_k\} \subseteq C \Rightarrow \{w_1, \ldots, w_k\} \subseteq \overline{A}$), let p be the greatest element of $\{1, \ldots, m-1\}$ such that $y_p \notin \{w_1, \ldots, w_k\}$, if such a p exists. If it does, then

$$y_m \in C \Rightarrow y_p \in \overline{C} \Rightarrow y_p \in A,$$

so let $s = \max(\{i : x_i = y_p\})$, and let $\varphi_2(x_1, \ldots, x_n) = s$. If $k, w_1, \ldots, w_k$ are found but p does not exist, then $y_m \in \overline{A} \Rightarrow \{y_1, \ldots, y_m\} \subseteq \overline{A}$, so let $\varphi_2(x_1, \ldots, x_n) = 0$. (Note that, regardless of the existence of p, $y_m \in \overline{A} \Rightarrow \varphi_2(x_1, \ldots, x_n) \downarrow= \mathrm{IMAX}_n^A(x_1, \ldots, x_n)$.) ∎

The following theorem is presented without proof, because the main part of the argument is very similar to the proof of Theorem 4.7.

Theorem 4.8 *$\overline{A}$ is retraceable iff there exists a partial recursive function φ such that $\mathrm{ran}(\varphi) \subseteq \{0, 1\}$ and $\mathrm{IMAX}_2^A(x_1, x_2) \in \{\varphi(x_1, x_2), 2\}$ for all x_1, x_2.*

Theorem 4.9 *A has a strong binary IMAX-limiter iff $\overline{A}$ is the union of a recursive set and a strongly retraceable set.*

Proof:

$\Rightarrow$: Suppose that A has a strong binary IMAX-limiter f. We can assume that $|D_{f(x,y)}| = 2$ for all x, y. Define a recursive set B by

$$B = \{y : (\exists x < y)[2 \notin D_{f(x,y)}]\}.$$

Then $B \subseteq \overline{A}$. Let $C = \overline{A} - B$. We can assume that C is nonempty. We define a (total) recursive function g that retraces A. Let c be the least element of C.

For $y \in \omega$, let $g(y)$ be the greatest number $x < y$ such that $x \notin B$ and $1 \notin D_{f(x,y)}$, if such an x exists; otherwise, let $g(y) = y$. Note that, for every y,

$$y \in C \Rightarrow (g(y) \in C \;\wedge\; g(y) \leq y \;\wedge\; (g(y) = y \text{ iff } y = c)).$$

Also, for all w, y,

$$(y \in C \;\wedge\; g(y) < w < y \;\wedge\; w \notin B) \Rightarrow D_{f(w,y)} = \{1, 2\} \Rightarrow w \in A \Rightarrow w \notin C.$$

Thus, for every $y \in C - \{c\}$, $g(y)$ is the greatest number $x < y$ such that $x \in C$; hence g retraces C.

$\Leftarrow$: Suppose that $\overline{A} = B \cup C$, where B is recursive and C is strongly retraceable. We can assume that $B \cap C = \emptyset$, since it is easily verified that $C - B$ is also strongly retraceable. Choose a (total) recursive retracing function f for C. We define a strong binary IMAX-limiter g for A.

Let $x, y \in \omega$. We can assume that $x \leq y$. If $x = y$, let $D_{g(x,y)} = \{0, 2\}$. If $x < y$ and $B \cap \{x, y\} \neq \emptyset$, let

$$D_{g(x,y)} = \begin{cases} \{0,2\}, & \text{if } x \in B; \\ \{0,1\}, & \text{if } x \notin B \ \wedge \ y \in B. \end{cases}$$

If $x < y$ and $\{x, y\} \subseteq \overline{B}$, determine whether or not there exist $k \geq 1$ and $w_1, \ldots, w_k \in \omega$ such that

- $w_k = y$,
- $f(w_1) = w_1$,
- for every i with $1 \leq i < k$, $f(w_{i+1}) = w_i$, and
- $k > 1 \Rightarrow w_1 < \cdots < w_k$.

If such $k, w_1, \ldots, w_k$ exist, let $S = \{w_1, \ldots, w_k\}$; otherwise, let $S = \emptyset$. Then let

$$D_{g(x,y)} = \begin{cases} \{0,2\}, & \text{if } x \in S; \\ \{1,2\}, & \text{otherwise.} \end{cases}$$

Note that $y \in A \Rightarrow \mathrm{IMAX}_2^A(x, y) = 2 \in D_{g(x,y)}$; moreover,

$$y \in \overline{A} \Rightarrow y \in C \Rightarrow (S \subseteq C \ \wedge \ S \neq \emptyset) \Rightarrow (x \in C \text{ iff } x \in S) \Rightarrow (x \in A \text{ iff } x \notin S).$$

Hence $\mathrm{IMAX}_2^A(x, y) \in D_{g(x,y)}$, regardless of whether $y \in A$. ▮

It is natural to conjecture that $\mathrm{IMAX}_2^A \in \mathrm{EN}(2)$ iff $\overline{A}$ is the union of a retraceable set and a recursive set, but we can prove only the "if" direction. Thus we do not know exactly which sets have binary IMAX-limiters.

We conclude by showing that it is impossible for a nonrecursive set to have both an IMAX-limiter and an IMIN-limiter.

Theorem 4.10 *If A has an n-ary IMAX-limiter and an m-ary IMIN-limiter for some $n, m \geq 1$, then A is recursive.*

Proof: Suppose that A has an n-ary IMAX-limiter g and an m-ary IMIN-limiter h for some $n, m \geq 1$. Without loss of generality, we can assume that $m \leq n$. We say that a node σ of the full binary tree $2^{<\omega}$ is *consistent with* g if, for all $x_1, \ldots, x_n \in \omega$ with $x_1 \leq \cdots \leq x_n < |\sigma|$, $\max(\{j : \sigma(x_j) = 1\}) \in W_{g(x_1,\ldots,x_n)}$, where we define $\max(\{j : \sigma(x_j) = 1\})$ to be 0 if $\sigma(x_1) = \cdots = \sigma(x_n) = 0$. Similarly, we define what it means for σ to be *consistent with* h.

Now let $\mathcal{T}$ consist of those nodes of $2^{<\omega}$ which are consistent with both g and h. Then $\mathcal{T}$ is an r.e. subtree of $2^{<\omega}$, and A is an infinite branch of $\mathcal{T}$. According to Lemma 2.7, if

B_{4^n-2} can be embedded in $\mathcal{T}$, then there exist natural numbers $x_1, \ldots, x_n$, nodes $\tau_1, \ldots, \tau_{n+1}$ of $\mathcal{T}$, and $b \in \{0,1\}$ such that, for i, j with $1 \leq i \leq n$ and $1 \leq j \leq n+1$,

$$\tau_j(x_i) = \begin{cases} 1-b, & \text{if } 1 \leq i < j; \\ b, & \text{if } j \leq i \leq n. \end{cases}$$

If $b = 0$, then one of the nodes $\tau_1, \ldots, \tau_{n+1}$ is not consistent with g (else $\{0, \ldots, n\} \subseteq W_{g(x_1,\ldots,x_n)}$, in contradiction to the assumption that $W_{g(x_1,\ldots,x_n)} \subset \{0, \ldots, n\}$). If $b = 1$, then, confining our attention to $x_1, \ldots, x_m$, one of the nodes $\tau_1, \ldots, \tau_{m+1}$ is not consistent with h. Therefore, B_{4^n-2} cannot be embedded in $\mathcal{T}$, and so, by the R.E. Tree Lemma, A is recursive.

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