Easy Parts of Quantum Graph Coloring

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Graph Coloring

**Notation** \([k] = \{1, \ldots, k\}\).
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Definition $G = (V, E)$ is $k$-colorable if there exists a mapping

$$COL: V \rightarrow [k]$$

such that $(\forall x, y \in V)[(x, y) \in E \implies COL(x) \neq COL(y)]$. 
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**Notation** The least \(k\) such that \(G\) is \(k\)-colorable is denoted \(\chi(G)\).
A Graph Coloring Game

Given $G$ and $k$ imagine the following game. A&B are on one team. They can communicate before the game but not during it. E is the other team.

1. E picks $x, y \in V$ at random (can have $x = y$).
2. A gets $x$, B gets $y$. They do not communicate.
3. A says $c_x \in [k]$; B says $c_y \in [k]$, simul. A&B are all powerful and can use random coins if they want. They can even use the same random coins.
4. 4.1 If $x = y$ and $c_x = c_y$, A&B win.
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How this Game Relates to Graph Coloring

If \( \chi(G) \leq k \) then A&B can WIN.

If \( \chi(G) \geq k + 1 \) then the probability that A&B win is < 1.

One could have defined \( \chi(G) \) in terms of this game.
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Are there any graphs where A&B can do better? I do not know.
What if A&B could Communicate?

If A&B could communicate $\lceil \log n \rceil$ bits then can reveal to each other which node they got, so A&B could win with prob 1, even if $k = 2$.

If A&B could communicate $a$ bits then might be able to increase their chance of winning (has not been looked at).

If A&B could share QUANTUM STUFF I DO NOT UNDERSTAND THAT INVOLVES ENTANGLEMENT then there are graphs $G$ with $\chi(G) = k$ such that A&B win with Prob 1 using $k' < k$. 

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One Case that is Known and Impressive

**Notation** if \( x, y \in \{0, 1\}^n \) then \( d(x, y) \) is the number of places they differ. This is also called the **Hamming Distance**.
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**Definition** The **Hadamard graph** $H_n$ is

$V = \{0, 1\}^n$ ($n$ is even)

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**Theorem**

1. \( \chi(H_n) = \Theta(2^n) \). (This is an old classical result.)
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**Theorem**

1. \( \chi(H_n) = \Theta(2^n) \). (This is an old classical result.)
2. If A&B can share **QUANTUM STUFF I DO NOT UNDERSTAND THAT INVOLVES ENTANGLEMENT** then A&B can win with prob 1 the Game with \( H_n \) and \( k = n \). So EXPONENTIAL improvement.
Easy Ramseyesque Theorem

A set is **homog** if every element has the same color.
A set is **rainbow** if every element has a different color.

**Known And Easy**
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- There exists a finite colorings of \{1, \ldots, 9\} with NO 4 homog and NO 4 rainbow.

```
1 2 3 4 5 6 7 8 9
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- For all finite colorings of \{1, \ldots, 10\} there will be either 4 homog or 4 rainbow.
First Question in Quantum Ramsey Theory

1. A, B, C, D are on one team. E is on the other.
2. E picks \(a, b, c, d \in [10]\) at random and gives \(a\) to A etc.
3. A, B, C, D each simul say a color in \([4]\).
4. 4.1 If all colors same or all colors different then A, B, C, D loses.
4.2 In any other case A & B win.

Classically A, B, C, D have prob \(< 1\) of winning.

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Future of Quantum Ramsey Theory

Look at other Theorems in Ramsey Theory and formulate Quantum Questions. Then answer them!
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