Linear Sorting (Chapter 8)

Note Title 11/8/2007

Comparison-based sorting problem has a worst-case lower bound of $\Omega(n \log n)$.

So to achieve better runtimes even in the worst case, we have to change the model.

Can we add more assumptions that an algorithm can use to do less work?

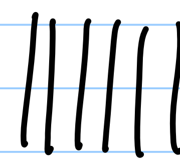
- Add more restrictions? (Recall that our sorts so far work on any comparable data)

- Could we have certain types of data where the runtime is vastly improved?

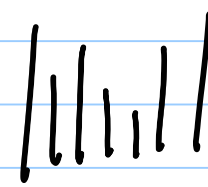
Huh? What do you mean, change the model?

Spaghetti Sort

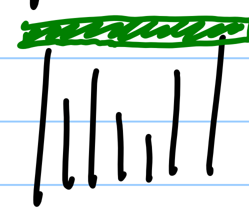
Take box of spaghetti.



Cut each spaghetti stick to the size of the sort key.



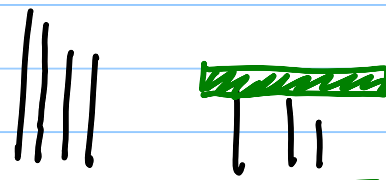
Take a piece of tape and tape it to the top of the highest stick.



Take the stick attached to the sticky tape and put aside.

Repeat, moving sticky tape down.

after a few iterations



sorted

to be sorted.

Memory "Sort" aka Hashtable } ON NON-NEG INTEGERS

- allocate array H[MAXINT]
- initialize all values in H to Ø
- go through input array A

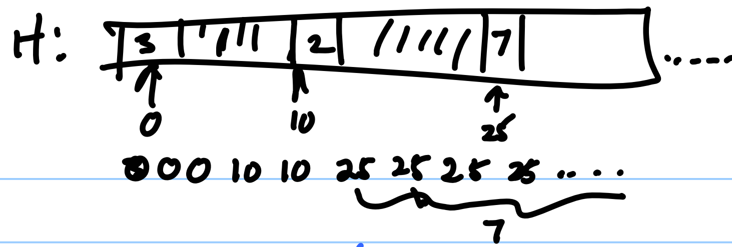
    if A[i] = k
    then H[k]++; } H[A[i]]++;

- go through H, print out values. } If a cell has value v>0, then print the cell index v times.

Side Note: Do we still require UNIQUE values for our input?

data comparisons ?

array reads ?



Memory Sort:

- Number of comparisons: None!

  (Is this correct? What about integer comparisons? Okay, so maybe $O(n)$.)

- Number of array reads?

  $O(n)$?

  $O(\text{MAXINT})$?

  - MAXINT is a constant — so $O(1)$.
  - Hey, wait a minute!!!

    (How big do you think MAXINT is wrt n?)

So is this supposed to be our promised Linear Sort??

Well... maybe not...

What are the issues with Memory Sort?

1. Not stable.

(Debugged in class - ignore previous slide that indicated stability.)

2. MAXINT.

Can we address these issues?

Counting Sort : Intuition

- Done on integers
- Values do not need to be unique
- Three arrays:

A:

2	5	3	0	2	3	0	3

input

B:

0	0	2	2	3	3	3	5

output

C:

2	0	2	3	0	1
0	1	2	3	4	5

← Scratch work array. Its size is based on the range of numbers in array A.

- Determine, for each input element x (in A), the number of elements less than x. This information can be used to place element x directly into its position in output array B.

  If 17 elements less than x, then x belongs in position 18. (more or less)

- What is C for? Count up how many occurrences of each input value. Then modify so each cell contains number of elements ≤ to the cell index:

2	2	4	7	7	8
0	1	2	3	4	5

Counting Sort.

// input must be nonnegative ints
// A is input, B is output

// for each input A[i], find
// all values less than it.

1. Find MaxVal = Max(A) ← $O(n)$
2. Initialize all C[0 .... MaxVal] to zero.
3. For j = 1 to n
   C[[A[j]] ++;
   // C[i] is # elements in A equal to i
4. For i = 1 to MaxVal
   C[i] = C[i] + C[i-1].
   // Now C[i] is # elements in A ≤ i

5. For k = n to 1
   Let val = A[k]. // value in A
   Let count = C[val]. // # elements ≤ val
   B[count] = val;
   C[val] = count - 1;

Counting Sort

What if our lowest value is $\gg 0$? (much greater)

OR : What if we have some negative integers?

Counting Sort

much greater

What if our lowest value is $\gg 0$?

OR: What if we have some negative integers?

We were already doing this.

- compute Max Val
- compute Min Val

We can get more precise — figure out range.

- give C size (Max − Min).
  C[0 . . . (Max − Min)].

- subtract min from all values
- do Counting Sort as before.

- add min to all elements in B.

alternatively, subtract min from values used as indices to C.

Counting Sort: Desirable Features.

Recall Vibha's last lecture:

DATA STABILITY means that items with the same keys stay in the same relative positions.

Why important?

- If satellite data carried around with element being sorted, we don't want to lose relative order. } We call the element a key.
- Sometimes we require stability for a sort because it is used as a subroutine for another sort that has this requirement. (We will see: RadixSort uses CountingSort)

- Unique keys - Not required.

- Are non-primitive keys allowed?

Runtime of Counting Sort (array reads, array writes)

$$\Theta(\max(n, \text{MaxVal}))$$

findMax

or

$$\Theta(\max(n, \underbrace{\text{range}}_{\text{MaxVal} - \text{MinVal} + 1}))$$

equivalent

$$= \Theta(n + \text{range})$$

if range is $O(n)$,
then algorithm is $\Theta(n)$.

- This sort is <u>good</u> for a large set of values in a small range. (Lots of duplicates.)
- Is it stable?
- Is it efficient for sorting SS numbers?

Radix Sort for nonneg. integers

Sort one digit at a time

329	→	457	→	329	→	329
457		657		839		457
657		329		457		657
839		839		657		839

What would happen if we sorted on most significant digit first?

sort rightmost digit first → sort leftmost digit last

Digit sorts must be STABLE.

Use Counting Sort on each colum

- MaxVal on digit sort is 9.

so digit sort is $O(\max(n,10)) \rightarrow O(n)$.

- So runtime of radix sort is $O(dn)$

$d = \#$ digits

Question about Format of Digits

What if we have different numbers of digits?

738
59
132
7
561

Need to "pad" with zeros: 738, 059, 132, 007, 561

A trivia moment...

What can this be used for?

For us old-timers...

- Used by card-sorting machines now found only in computer museums

- Sorter mechanically "programmed" to examine a given column of each card and distribute in one of 12 bins depending on where punched.

General Radix Sort Runtime

d = # "digits"
(could be other data)

r = range of each digit

n = # values.

Only "d" passes are required.

Radix sort runtime is $O(d(n+r))$

(equivalent to $O(d \cdot \max(n, r))$)

if d is fixed and $r \in O(n)$,
then radix sort is linear.

Question 1: If we have n b-bit integers, can we sort them in $\Theta(b \cdot n)$ time?

Yes, trivially:

$d = b$ (# of bits)

$r = 2$ (0, 1)

$n = n$

$\Longrightarrow$ $\boxed{\begin{array}{l}\Theta(d(n+r)) \\ \Theta(b(n+2)) \\ \Theta(bn)\end{array}}$

Question 2: How many bits are used to represent the numbers in the range $0 \ldots n-1$?

$\log_2 n$, so $\Theta(n \log n)$ sort!

Question 3: What if we group the bits into clusters of size $\underline{\underline{r}}$?

#digits

$$\text{Radix Sort} \in \Theta\left(\frac{b}{r}\left(n + 2^r\right)\right)$$

Counting range

Tradeoff is Speed vs. Memory used in Counting Sort Part

Claim: Given the number of bits to represent n numbers is $O(\log n)$, then if we do all of the bits in a single grouping, RadixSort runs in $\Theta(n)$ time.

$b \in O(\log n)$

let $r = \log_2 n$ (ie., # of bits)

RadixSort is $\Theta\left(\frac{b}{r}\left(n + 2^r\right)\right)$ $\quad \frac{b}{r} = 1$

$\Theta(n + 2^r)$

$\Theta(n + 2^{\log_2 n})$

$\Theta(n + n)$

$\Theta(n)$

Lots of hidden constants + memory.

Thought Question

Can we sort n values that are in the range $0 \ldots n^2$ in $O(n)$ time?

Can we sort n values that are in the range $\emptyset \ldots n^2$ in $O(n)$ time?

- n values in range 0 to $n \times n$.
- Can we take each value and "cut it in half"?

$\Theta(d(n+r))$ $d=2$

$\Theta(2(n+n))$ $n=n$

$r=0$ to n

$\Theta(n)$

$\Rightarrow$ Hides work/memory for internal stable sort