Normal / Truth-teller Problem

You have
\[ T \text{ people who always tell the truth}. \quad (\text{Truth-teller}) \]
\[ N \text{ people who may or may not tell the truth}. \quad (\text{Normal}) \]

They know who is who.
\[ T+N \text{ is even}. \]
\[ T > N \]

Give an algorithm that finds all the truth-tellers.
Algorithm.

1. Ask everyone if they are a truth-teller.
   Split into yes[1,...,p] and no[1,...,q]

Anyone who says no is Normal.
So no[1,...,q] ⊆ Normal.

2. For all those who say yes.
   ask if all others in yes[1,...,p]
   are truth-tellers.

Then for each person you have p opinions.

```
person1: T| l | l | l | l | l | l | l | l | T
person2: T| l | l | l | l | l | l | l | l | T
person p: T| l | l | l | l | l | l | l | l | T
```

p opinions
If \( T > N \), so if more no's for person \( i \) then person \( i \) is Normal.

If more yes responses for person \( i \) that person \( i \) is a Truth-teller.

\[
\text{\# questions} = T + N + (p)(p - 1)
\]

In worst case
\[ q = \emptyset \]
\[ p = T + N. \]

So \( \text{\# questions} = (T + N) + (T + N)(T + N - 1) \)

\[ = O(T + N)^2 \]
This problem can also be solved with Divide and Conquer.

1. Split into pairs.
2. Ask P1 about P2.
   Ask P2 about P1.

<table>
<thead>
<tr>
<th>Case</th>
<th>P1 sez</th>
<th>P2 sez</th>
<th>Can both be Tru?</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>P2 is Nor</td>
<td>P1 is Nor</td>
<td>no</td>
</tr>
<tr>
<td>II</td>
<td>P2 is Nor</td>
<td>P1 is Tru</td>
<td>no</td>
</tr>
<tr>
<td>III</td>
<td>P2 is Tru</td>
<td>P1 is Nor</td>
<td>no</td>
</tr>
<tr>
<td>IV</td>
<td>P2 is Tru</td>
<td>P1 is Tru</td>
<td>yes</td>
</tr>
</tbody>
</table>

If both are truth-tellers, then both must call the other Tru.

Note that in case IV, it’s also possible that both are Normal.
Please note:

\[ T > N, \text{ and } T+N \text{ is even.} \]

Let's look at some examples.

\begin{align*}
\text{ex: } T &= 3 \\
N &= 1 \\
\text{Tru Tru Tru Nor} &\quad \text{Tru Tru Tru Tru Nor Nor}
\end{align*}

always 2 more Tru's than Nor's,
so, always at least one

Case IV where both persons are Truth-tellers.

3) so Take all Case IV pairs.
Drop one person from each pair.

4) RECURSE on who's left.

until you have only 2 left \( \rightarrow \) they must both be truth-tellers.
Example: $T = 6 \ N = 2$

1. 4 pairs in case IV.

2. Drop one element from each case IV pair. Note: this won’t change
   \[ \# \text{Tru} > \# \text{Nor}. \]

3. Recurse.
   at most 1 case IV pair.

\[ P1, P3 \]
say P1 and P3 are the Case IV pair.

They must both be truth-tellers.

⑤ Ask P1 or P3 about all the others.
Worst Case Performance

of Divide and Conquer Algorithm.

worst case when all pairs in Case IV.
(ignore step #5 for now)

\[ T(n) = T(n + N) + T\left(\frac{n}{2}\right) \]

first question to pairs

if all pairs in Case 4.

\[ n = n + N \]

\[ T(n) = n + T\left(\frac{n}{2}\right) \]

Apply Master Theorem.

\[ a = 1 \quad f(n) = n \]
\[ b = 2 \]
\[ \log_b a = \log_2 1 = 0 \]
\[ n^0 < n \]

Case III holds

\[ \Theta(n) \]

\[ T(2) = 0 \]

\[ T(n) = n + T\left(\frac{n}{2}\right) \]

Prove \( T(n) \in O(n) \) using constructive induction.

IH: \( 4 \leq i < n \quad T(i) \leq ci \)

\[ T(n) \leq cn \]

\[ n + T\left(\frac{n}{2}\right) \leq cn \]

\[ n + \frac{cn}{2} \leq cn \text{ by IH.} \]

\[ \left(\frac{c}{2} + 1\right) n \leq cn \]
\[ (\frac{c}{2} + 1)n - cn \leq 0 \]
\[ (\frac{c}{2} + 1 - c)n \leq 0 \]
\[ (1 - \frac{c}{2})n \leq 0 \]

True when \[ 1 - \frac{c}{2} \leq 0 \]

\[ \frac{c}{2} \geq 1 \]

\[ c \geq 2 \text{ (let)} \]  
\[ n_0 = 4. \]

Base case:  
\[ T(4) = 4, \text{ and } 4 \leq 8. \]
Recurrence "Tree" Method

\[ T(n) = n + T\left(\frac{n}{2}\right) \]

Level 0

\[ n \]

Level 1

\[ \frac{n}{2} \]

Level 2

\[ \frac{n}{4} \]

Level \( m \)

\[ 2 \]

\[ m = \log_2 n - 1 \]

\[ \sum_{i=0}^{\log_2 n - 2} \frac{n}{2^i} = n \sum_{i=0}^{\log_2 n - 2} \left(\frac{1}{2}\right)^i \]
\[ \leq n \sum_{i=0}^{\infty} \left( \frac{1}{2} \right)^i \]

\[ = n \left( \frac{1}{1-\frac{1}{2}} \right) \]

\[ T(n) \leq 2n \]

\[ T(n) \in O(n) \]