Normal / Truthteller Problem

Note Title 10/3/2007

You have

T people who always tell the truth. (Truthtellers)

N people who may or may not tell the truth. (Normal)

They know who is who.

$T+N$ is even.

$T > N$

Give an algorithm that finds all the truthtellers.

Algorithm.

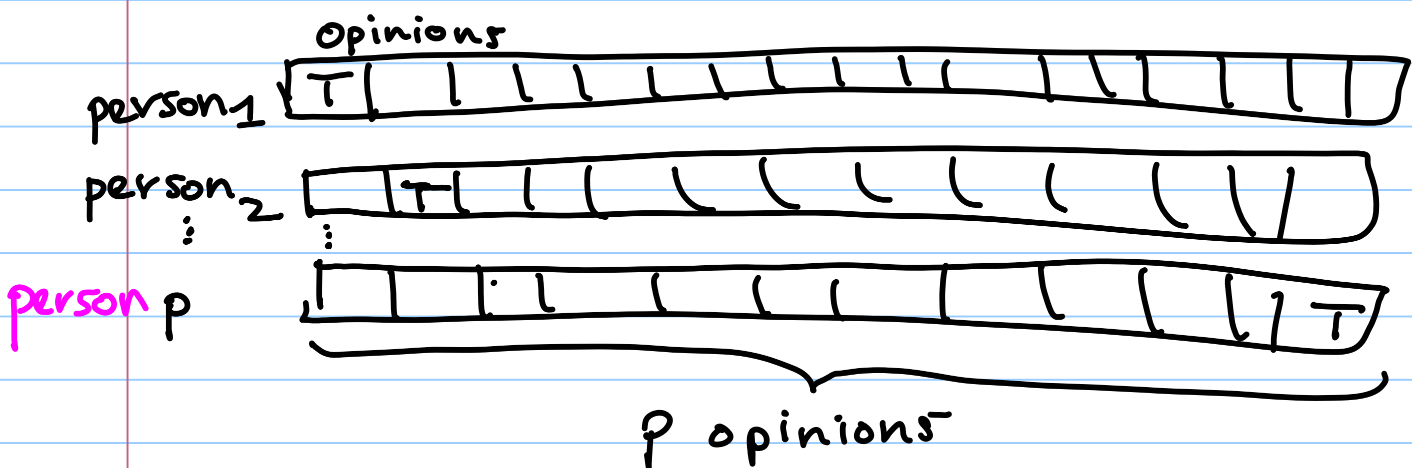
1. Ask everyone if they are a truthteller.
   Split into yes[1....p] and no[1.....q]

Anyone who says no is Normal.
So no[1... q] ← Normal.

2. For all those who say yes -

   ask if all others in yes[1...p] are truthtellers.

   Then for each person you have p opinions.



$T > N$, so if more no's for person$_i$ then person$_i$ is Normal.

If more yes responses for person$_i$ that person$_i$ is a Truthteller.

questions

$$= T+N + (p)(p-1)$$

in worst case

$$q = \emptyset$$
$$p = T+N.$$

so # questions is $(T+N) + (T+N)(T+N-1)$

$$= O(T+N)^2$$

This problem can also be solved with Divide and Conquer.

1. Split into pairs.
2. Ask P1 about P2.
   Ask P2 about P1.

Case	P1 sez	P2 sez	Can both be Tru?
I	P2 is Nor	P1 is Nor	no
II	P2 is Nor	P1 is Tru	no
III	P2 is Tru	P1 is Nor	no
IV	P2 is Tru	P1 is Tru	yes

If both are truthtellers, then both must call the other Tru.

Note that in case IV, it's also possible that both are Normal.

Please note:

$T > N$, and $T+N$ is even.
Let's look at some examples.

ex: $T = 3$ $N = 1$ Tru Tru Tru Nor	$T = 4$ $N = 2$ Tru Tru Tru Tru Nor Nor

<u>always</u> 2 more Tru's than Nor's,

so, always at least one

Case IV where both persons are Truthtellers.

(3) so Take all Case IV pairs.
Drop one person from each pair.

(4) RECURSE on who's left.

until you have only 2 left →
they must both be truthtellers.

example: $T=6$ $N=2$

P1	P2	P3	P4	P5	P6	P7	P8
Tru	Tru	Tru	Tru	Tru	Tru	Nor	Nor
sez P1 is Tru	sez P2 is Tru	sez P4 is Tru	sez P3 is Tru	sez P6 is Tru	sez P5 is Tru	sez P8 is Tru	sez P7 is Tru

(1-2) 4 pairs in Case IV.

(3) Drop one element from each Case IV pair. Note: this won't change # Trus > # Nor.

P1	P3
Tru	Tru

P5	P7
Tru	Nor

(4) Recurse.

at most 1 Case IV pair.

P1, P3

say P1 and P3 are the Case IV pair.

They must both be truthtellers.

⑤ Ask P1 or P3 about all the others.

Worst Case Performance

of Divide and Conquer Algorithm.

worst case when all pairs in Case IV.
(ignore step #5 for now)

$$T(n) = \underbrace{T+N}_{\text{first question to pairs}} + T\left(\frac{T+N}{2}\right)$$

if all pairs in Case 4.

$n = T+N$

$$T(n) = n + T\left(\frac{n}{2}\right)$$

Apply Master Theorem.

$a = 1$ $\quad$ $f(n) = n$

$b = 2$

$\log_b a = \log_2 1 = 0$

$$n^0 < n$$

Case III holds

$$\Theta(n)$$

$T(2) = 0$

$$T(n) = n + T\left(\frac{n}{2}\right)$$

Prove $T(n) \in O(n)$ using constructive induction.

IH: $4 \leq i < n \quad T(i) \leq ci$

$$T(n) \leq cn$$

$$n + T\left(\frac{n}{2}\right) \leq cn$$

$$n + \frac{cn}{2} \leq cn \quad \text{by IH.}$$

$$\left(\frac{c}{2} + 1\right) n \leq cn$$

$$\left(\frac{c}{2}+1\right)n - cn \leq 0$$

$$\left(\frac{c}{2}+1-c\right)n \leq 0$$

$$\left(1-\frac{c}{2}\right)n \leq 0$$

true when $1-\frac{c}{2} \leq 0$

$$\frac{c}{2} \geq 1$$

$c \geq 2$ let: $c = 2$

$n_0 = 4$.

Base case:
$T(4) = 4$, and $4 \leq 8$. ✓

Recurrence "Tree" Method

$T(n) = n + T\left(\frac{n}{2}\right)$

Level ϕ $\boxed{n}$

Level 1 $\boxed{n/2}$

Level 2 $\boxed{n/4}$

$\vdots$

Level m $\boxed{2}$

$m = \log_2 n - 1$

$$\sum_{i=0}^{\log_2 n - 2} \frac{n}{2^i} = n \sum_{i=0}^{\log_2 n - 2} \left(\frac{1}{2}\right)^i$$

$$\le n \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i$$

$$= n \left(\frac{1}{1 - \frac{1}{2}} \right)$$

$$T(n) \le 2n$$

$$T(n) \in O(n)$$