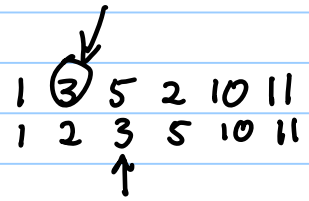
Order Statistics (Selection Problems)

Note Title 10/1/2007

Simply stated: "Given a list of *n* unique values, find the i^{th} smallest."

Common Examples

- 1^{st} smallest (Minimum)
- n^{th} smallest (Maximum)
- $\lceil n/2^{th} \rceil$ smallest (Median) → 

How can we approach solving such problems?

Trivial Way: Sort the list and then return the i^{th} position.

$n \log n + i$ e.g., Mergesort

This is clearly not a good approach for things such as minimum and maximum.

This may or may not be a good approach for other problems such as median finding.

Minimum

- In the worst case, finding the minimum requires n-1 comparisons.
- Finding the minimum can easily be done using at worst n-1 comparisons:
  - Call the first item in the list the smallest.
  - For each item remaining, compare it to the item currently considered smallest and if it is smaller than that item, set this new item as the smallest.
- Do other algorithms exist? Sure, but are they better? What would the runtime be of the following recursive algorithm?
  - Split the list in half.
  - Find the minimum of each half.
  - Take the minimum of the two "local" minima returned.

Assume n is a power of 2.

$T(1) = 0$

$T(n) \stackrel{?}{=} 2T(\frac{n}{2}) + 1$

$T(n) \stackrel{?}{=} n-1$

Base: $n=1$ $\quad T(1) = 0 \stackrel{?}{=} n-1 = 1-1$ ✓ $\quad n=2$ $\quad T(2) = 1 \stackrel{?}{=} n-1 = 2-1$ ✓

→ IH: $\forall i < k \quad T(i) = i-1$

IS: $T(k) \stackrel{?}{=} k-1$

$2T(\frac{k}{2}) + 1 \stackrel{?}{=} k-1$

$2(\frac{k}{2} - 1) + 1 \stackrel{?}{=} k-1$

$k-2+1 \stackrel{?}{=} k-1$

$k-1 = k-1$ ✓

Maximum

Is there any practical difference between algorithms for finding the maximum as opposed to finding the minimum value in a list?

Minimum AND Maximum

Consider the following scenario:

You are given a list of coordinates and are asking to return a bounding box for these points.

Your getBoundingBox() method would need to find both the minimum x-coordinate as well as the maximum x-coordinate (and then do the same for the y-coordinates).

In general, given a list of items, it is easy to find the minimum and the maximum using 2(n-1) comparisons.

Can we do better?

Min/Max Algorithm #1

What is the runtime of the following algorithm to find the minimum and maximum "at the same time" and will it always give the correct results?

- Traverse the list once, two at a time, comparing pairs.
- As this is done, create two sub-lists: SubList1 for the greater of the pair-wise comparisons and SubList2 for the lesser.
- Call the regular maximum algorithm on SubList1 and the regular minimum algorithm on SubList2.

$\{x_1\ x_2\ x_3\ x_4\ x_5\ x_6 \dots\}$

Stage 1: $\frac{n}{2}$ + Stage 2: $(\frac{n}{2}) - 1$ + $(\frac{n}{2}) - 1$

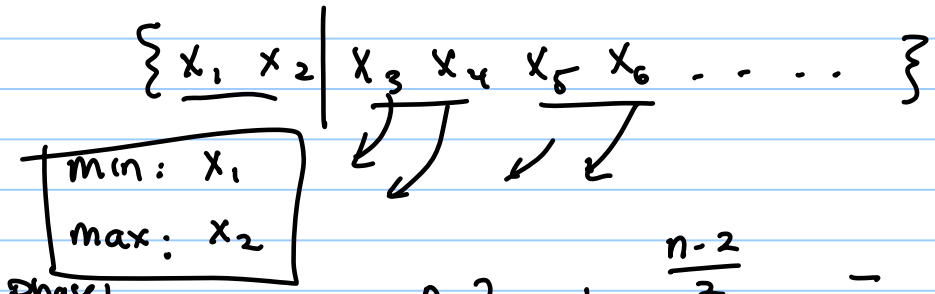
$3(\frac{n}{2}) - 2 = \boxed{\frac{3n}{2} - 2}$

$2(n-1) = 2n - 2$

Min/Max Algorithm #2

What is the runtime of the following algorithm to find the minimum and maximum "at the same time" and will it always give the correct results?

- Compare the first two elements in the list. Set the smaller as min and the larger as max. } Phase 1
- For the remaining elements of the list:
  - Pair up and compare the items in each pair. } Phase 2
  - Compare the smaller of the pair to the current min, replacing it if we have a new min.
  - Compare the larger of the pair to the current max, replacing it if we have a new max. } Phase 3



$$1 + \frac{n-2}{2} + \frac{\frac{n-2}{2}}{\frac{n-2}{2}} = \frac{3n}{2} - 2$$