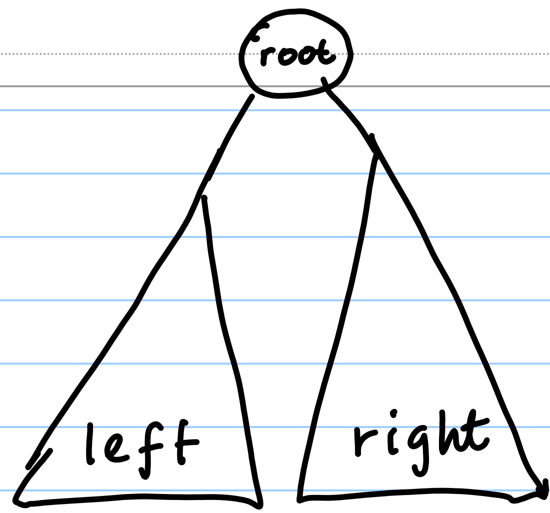
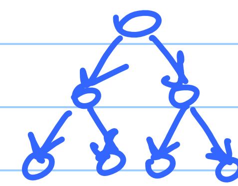
10/31/2007

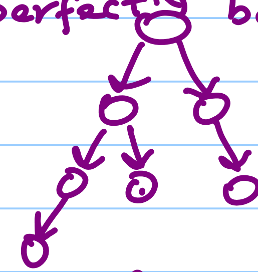
Does AVL rule guarantee height $O(\log n)$?



fully balanced tree: $h = \log_2 n$



but AVL tree?
(not perfectly balanced)



is $h \in O(\log n)$?

Let # nodes in left subtree $= n_{left}$
nodes in right subtree $= n_{right}$

$$n_{total} = n_{left} + n_{right} + 1$$

1. Because left and right subtree are AVL trees,

$$height_{left} = height_{right} \pm \{0, 1\}$$

worst case is when they are not even,
so let $height_{right} = height_{left} - 1$

or $h_{right} = h_{left} - 1$.

2. $h_{total} = h_{left} + 1$

root has height h, left subtree has height $h-1$, and right subtree has height $h-2$.

3. So...

$$n = n_{left} + n_{right} + 1$$

can be written as $n_h = n_{h-1} + n_{h-2} + 1$

Note Title 11/7/2007

Is worst case height of an AVL tree $O(\log n)$?

we know:

$$n_h \geq n_{h-1} + n_{h-2} + 1$$

n_h = # nodes in a tree of height h

we want to prove

$$h \leq \log n_h$$

This is equivalent to:

$$h \leq \log_x n_h \quad , x > 1$$

$$x^h \leq (x)^{\log_x n_h}$$

$$x^h \leq n_h$$

$n_h^{\log_x x} = n_h^1$ $\log_x x \rightarrow 1$

So if we prove that

$$x^h \leq n_h$$

then we have proven that height is $O(\log n)$.

Prove $x^h \leq n_h$ using constructive induction.

Base case.

$h = \emptyset$, $n_h = 1$

$x^0 \leq 1$ ✓.

Inductive Hypothesis

$\forall i \; 0 \le i < h, \; x^i \le n_i$

Inductive Step.

$$x^h \le n_h$$

$n_h \ge n_{h-1} + n_{h-2} + 1$

so if $x^h \le n_{h-1} + n_{h-2} + 1$,

then goal is proven.

new goal:

$$x^k \leq n_{k-1} + n_{k-2} + 1$$

apply IH.

if $x^k \leq x^{k-1} + x^{k-2} + 1$,
then goal is proven.

new goal:

$$x^k \leq x^{k-1} + x^{k-2} + 1$$

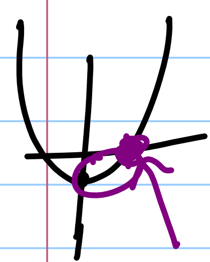
$$x^2 \leq x + 1 + \frac{1}{x^{k-2}}$$

if $x^2 \leq x+1$,
then goal is proven.

new goal:

$$x^2 \leq x+1$$

$$x^2 - x - 1 \leq 0$$



$$x^2 - x - 1 = 0$$

Let $x = \dfrac{1 + \sqrt{1 - 4(1)(-1)}}{2}$

$$x = \frac{1+\sqrt{5}}{2} = \phi$$

Could we use an AVL tree
to sort?