1. What is the upper bound for comparison-based sorting?
   
   notes:
   - quicksort \( O(n^2) \) in worst case
   - insertion sort \( O(n) \) in best case
   - merge sort \( O(n \log n) \) in worst case
   - upper bound is \( n \log n \)

2. So what is the lower bound?

   Use the same strategy as before—
   
   generic decision tree
Generic Decision Tree

For Comparison-Based Sort

1. Assume all input elements are distinct. (makes analysis easier)

2. Then, comparisons can be done with '<'.

3. Model all comparison-based sorts as follows:
   - Input is $a_1, a_2, a_3, \ldots, a_n$
   - Each node in tree is a comparison
   - Path through tree describes comparisons done in one execution of algorithm
   - Leaves are all possible outputs
all possible sorted outputs

if input is $a_1, a_2, a_3, a_4$ (say 4378)
output is $a_2, a_3, a_4$ (3478)
path is --- in blue
first comparison

second comparison

third comparison

\[ a_1 a_2 a_3 \ldots a_n \quad a_2 a_3 a_4 \ldots a_n \quad a_3 a_4 a_5 \ldots a_n \]

all possible sorted outputs

unique path to all outputs

How Many Outputs are there?
How Many Outputs are there?

Given input $a_1, a_2, \ldots, a_n$

Possible outputs of a sort algorithm are:

$$\left\{ \begin{array}{c}
    a_1, a_2, \ldots, a_n \\
    a_2, a_1, \ldots, a_n \\
    a_3, a_1, \ldots, a_n \\
    \vdots \\
    a_n, a_{n-1}, \ldots, a_1 \\
\end{array} \right.$$ 

how many?

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$$

↑ possibilities for first value
↑ possibilities for second value

$n!$ possible outputs
leaves = all possible sorted outputs

# leaves = n!

max # comparisons in a path = \( \log_2(n!) \) (#levels - 1)

Lower bound of \( \Theta(\log(n!)) \) for sorting
So in worst case, sorting in herently takes $\log_2(n!)$ comparisons!

How does that compare to the upper bound?
\[ \log(n!) \in O(n \log n) \]

\[ \log(n!) \leq cn \log n \]

Note: \( n! < n^n \)

\[ \frac{n(n-1)(n-2) \ldots 1}{n \cdot n \cdot n \cdot n \cdot n} \]

so

\[ \log(n!) < \log(n^n) \]

New goal:

\[ \log(n^n) \leq cn \log n \]

\[ n \log n \leq cn \log n \]

Let \( c = 1, \ n_0 = 1. \)
\[ \log(n!) \in \Omega(n \log n) \]

\[ c n \log n \leq \log(n!) \]

\[ c n \log n \leq \log[(n)(n-1)\ldots 1] \]

\[ \log AB = \log A + \log B \]

\[ c n \log n \leq \log n + \log n-1 + \ldots \log 1 \]

\[ c n \log n \leq \sum_{i=2}^{n} \log i \]

\[ \int_{1}^{n} \log i \, di = \left[ \log i \right]_{1}^{n} - \int_{1}^{n} 1 \, di \]

\[ = n \log n - n + 1 \]

So new goal:

\[ c n \log n \leq n \log n - n + 1 \]
\[ cn \log_e n \leq n \log_e n - n + 1 \]

New goal:

\[ cn \log_e n \leq n \log_e n - n \]

\[ c \log_e n \leq \log_e n - 1 \]

\[ (c-1) \log_e n \leq -1 \]

Need \( c < 1 \)

Let \( c = \frac{1}{2} \)

\[-\frac{1}{2} \log_e n \leq -1 \]

\[ \log_2 n \geq 2 \log_2 e \]

\[ n \geq 2^{2 \log_2 e} = (2^{\log_2 e})^2 = e^2 \approx 7.39 \]

Let \( n_0 = 8 \)

So \( \log(n!) \in \Theta(n \log n) \)