Application of Master Theorem

Note Title 9/28/2007

$T(n) = 2T\left(\frac{n}{3}\right) + n$

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

$a = 2,\ b = 3,\ f(n) = n$

Need to compare $f(n) = n$ to $n^{\log_3 2} \approx n^{.63}$

Which of these cases apply?

Case I : $f(n) = n \overset{?}{\in} O(n^{.63-\epsilon})$
Case II : $f(n) = n \overset{?}{\in} \Theta(n^{.63})$
Case III : $f(n) = n \overset{?}{\in} \Omega(n^{.63+\epsilon})$

Case III applies.

So what asymptotic bound applies?

$\boxed{\begin{array}{l} \text{Case I}: T(n) \in \Theta(n^{\log_b a}) \\ \text{Case II}: T(n) \in \Theta(n^{\log_b a} \log n) \\ \text{Case III}: T(n) \in \Theta(f(n)) \\ \quad a f\left(\frac{n}{b}\right) \leq c f(n),\ c < 1 \end{array}}$

$\Theta(f(n)) = \Theta(n)$

$2 \cdot \frac{n}{3} \leq c \cdot n \checkmark \ (c = \frac{2}{3} < 1)$

So $T(n) \in \Theta(n)$

$T(n) = 3T\left(\frac{3n}{4}\right) + n$

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

$a = 3,\ b = \frac{4}{3},\ f(n) = n$

Need to compare $f(n) = n$ to $n^{\log_{\frac{4}{3}} 3} \approx n^{3.8}$

Which of these cases apply?

Case I : $f(n) = n \overset{?}{\in} O(n^{3.8-\epsilon})$
Case II : $f(n) = n \overset{?}{\in} \Theta(n^{3.8})$
Case III : $f(n) = n \overset{?}{\in} \Omega(n^{3.8+\epsilon})$

Case I applies.

So what asymptotic bound applies?

$$\boxed{\begin{array}{l} \text{Case I}: T(n) \in \Theta(n^{\log_b a}) \\ \text{Case II}: T(n) \in \Theta(n^{\log_b a} \log n) \\ \text{Case III}: T(n) \in \Theta(f(n)) \\ \quad a f\left(\frac{n}{b}\right) \le c f(n),\ c < 1 \end{array}}$$

$$\downarrow$$

$\Theta(n^{\log_{\frac{4}{3}} 3}) \approx \Theta(n^{3.8})$

So $T(n) \in \Theta(n^{\log_{\frac{4}{3}} 3}) \approx \Theta(n^{3.8})$

$T(n) = 2T(\frac{n}{2}) + n$

$T(n) = aT(\frac{n}{b}) + f(n)$

$a = 2,\ b = 2,\ f(n) = n$

Need to compare $f(n) = n$ to $n^{\log_2 2} = n^1$

Which of these cases apply?

Case I : $f(n) = n^1 \overset{?}{\in} O(n^{1-\epsilon})$
Case II : $f(n) = n^1 \overset{?}{\in} \Theta(n)$
Case III : $f(n) = n^1 \overset{?}{\in} \Omega(n^{1+\epsilon})$

Case II applies.

So what asymptotic bound applies?

$$\boxed{\begin{array}{l} \text{Case I}: T(n) \in \Theta(n^{\log_b a}) \\ \text{Case II}: T(n) \in \Theta(n^{\log_b a} \log n) \\ \text{Case III}: T(n) \in \Theta(f(n)) \\ \quad af(\frac{n}{b}) \le cf(n),\ c < 1 \end{array}}$$

$\downarrow$

$\Theta(n^1 \log n)$

So $T(n) \in \Theta(n \log n)$

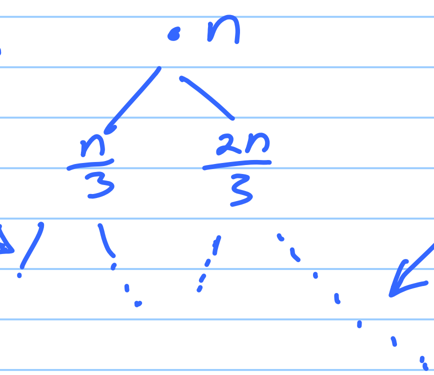
What algorithm has a runtime of $T(n) = 2T(\frac{n}{2}) + n$?

Can we apply the Master Theorem to

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n \quad ?$$

NO!? Why not? Master Theorem requires a balanced tree. This is not balanced.

This is the shortest path.



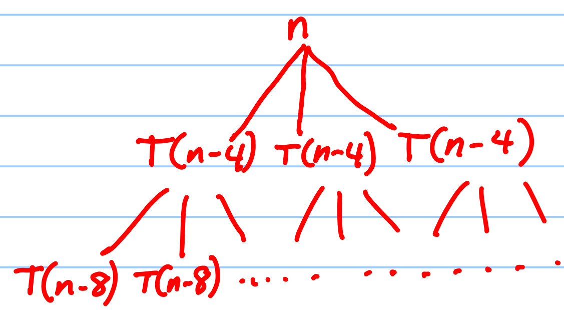
This is the longest path.

Can we apply the Master Theorem to

$T(n) = 3T(n-4) + n$?

NO!? Why not? Isn't the tree balanced? Yes! So Master theorem should apply! $\longrightarrow$ NO. There is an "a" but "$n-4$" is not in the form "$\frac{n}{b}$".

Must resort to tree drawing.



$n \qquad = 3^0 \cdot n - 3^0 \cdot 0 \cdot 4$

$3n - 3 \cdot 4 \qquad = 3^1 \cdot n - 3^1 \cdot 1 \cdot 4$

$9n - 9 \cdot 8 \qquad = 3^2 \cdot n - 3^2 \cdot 2 \cdot 4$

Levels $= \lceil \frac{n}{4} \rceil$

Amt of work per level $(\ell) = 3^\ell \cdot n - 3^\ell \cdot \ell \cdot 4 \approx 3^\ell \cdot n$

$$\sum_{i=0}^{n/4} 3^i \cdot n - \sum_{i=0}^{n/4} 3^i \cdot i \cdot 4 = n \sum_{i=0}^{n/4} 3^i - 4 \sum_{i=0}^{n/4} 3^i \cdot i$$

$n \cdot \frac{3^{n/4+1} - 1}{2}$

Integral approx. $\int 3^x \cdot x$