Recursion Unrolling

Note Title 10/25/2007

$$T(n) = \frac{2}{n} \sum_{pos_p=0}^{n-1} T(pos_p) + (n-1)$$

$$T(n) = \frac{2}{n} \sum_{i=0}^{n-1} T(i) + \Theta(n)$$

$T(\emptyset) = \emptyset$, so

$$T(n) = \frac{2}{n} \sum_{i=1}^{n-1} T(i) + an$$

idea behind unrolling is to represent

$T(n)$ in terms a sensible smaller input size.
(in our case, $T(n-1)$)

$$T(n) = \frac{2}{n}\left[T(1) + T(2) + \ldots T(n-1)\right] + an$$

Step 1: Rewrite $T(n)$ only in terms of $T(n-1)$ by SUBTRACTING:

Get rid of n in the denominator.

$$nT(n) = 2\sum_{i=1}^{n-1} T(i) + an^2$$

$$T(n-1) = \frac{2}{\cancel{n-1}}\sum_{i=1}^{n-2} T(i) + a(n-1)$$

$$(n-1)T(n-1) = 2\sum_{i=1}^{n-2} T(i) + a(n-1)^2$$

$$nT(n) - (n-1)T(n-1) = 2T(n-1) + an^2 - a(n-1)^2$$

$$nT(n) - (n-1)T(n-1) = 2T(n-1) + a(n^2 - (n-1)^2)$$

$$
\begin{aligned}
nT(n) - (n-1)T(n-1) &= 2T(n-1) + a(n^2 - (n-1)^2) \\
&= 2T(n-1) + a(n^2 - n^2 + 2n - 1) \\
&= 2T(n-1) + a(2n-1)
\end{aligned}
$$

$$nT(n) = (n-1)T(n-1) + 2T(n-1) + a(2n-1)$$

$$nT(n) = (n+1)T(n-1) + a(2n-1)$$

Step 2: Recognize a pattern.

$$\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + \frac{a(2n-1)}{n(n+1)}$$

same form

$$\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + \frac{a(2n-1)}{n(n+1)}$$

same form

step 2.5: Simplify messy stuff:

$$\frac{T(n)}{n+1} \leq \frac{T(n-1)}{n} + \frac{2a}{n}$$

Call the pattern Temp.

$$\text{Temp}(x) = \frac{T(x)}{x+1} \qquad (\text{Temp}(x))(x+1) = T(x)$$

$$\text{Temp}(n) \leq \text{Temp}(n-1) + \frac{2a}{n}$$

$$Temp(n) \le Temp(n-1) + \frac{2a}{n}$$

Step 3: Unroll

$$Temp(n-1) \le Temp(n-2) + \frac{2a}{n-1}$$

$$Temp(n) \le \underbrace{Temp(n-2) + \frac{2a}{n-1}}_{Temp(n-1)} + \frac{2a}{n}$$

$$Temp(n-2) \le Temp(n-3) + \frac{2a}{n-2}$$

$$Temp(n) \le Temp(n-3) + \frac{2a}{n-2} + \frac{2a}{n-1} + \frac{2a}{n}$$

$$Temp(n) \le \frac{2a}{1} + \frac{2a}{2} + \dots + \frac{2a}{n-1} + \frac{2a}{n}$$

$$\text{Temp}(n) \le \frac{2a}{1} + \frac{2a}{2} + \ldots + \frac{2a}{n-1} + \frac{2a}{n}$$

Step 4: simplify.

$$\text{Temp}(n) \le \sum_{i=1}^{n} \frac{2a}{i} = 2a \sum_{i=1}^{n} 1/i$$

$$\le 2a \int_{1}^{n+1} 1/i \, di$$

$$= 2a \cdot \ln i \Big|_{1}^{n+1}$$

$$\text{Temp}(n) \le 2a \ln(n+1)$$

Step 5: Go back to $T(n)$

$$\frac{T(n)}{n+1} \le 2a \ln(n+1)$$

$$\frac{T(n)}{n+1} \leq 2a \ln(n+1)$$

$$T(n) \leq (n+1)(2a)(\ln(n+1))$$

$$\boxed{T(n) \in O(n \ln n)}$$