Last update: January 28, 2010

Problem solving and search

CMSC 421: Chapter 3

Motivation and Outline

Lots of AI problem-solving requires trial-and-error search
Chapter 3 describes some algorithms for this

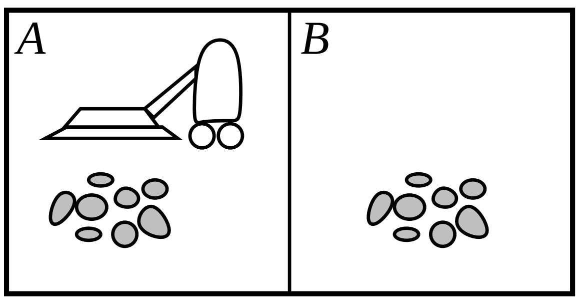
- ♢ Types of problems and agents
- ♢ Problem formulation
- ♢ Example problems
- ♢ Basic search algorithms

Problem types

Deterministic, fully observable $\Longrightarrow$ *classical search problem*

- ♢ agent knows exactly which state it starts in, what each action does
- ♢ no exogenous events (or else they're encoded into the actions' effects)
- ♢ solution is a sequence, can predict future states exactly

E.g., Vacuum World with **no** exogenous events
(hence, rooms won't spontaneously get dirty again)

Initial state: 

Goal: have both rooms clean

Solution: $[Suck, Right, Suck]$

Problem types

Non-observable

- ♢ Agent may have no idea where it is
- ♢ solution (if any) is a sequence that is *conformant*, i.e., guaranteed to work under all conditions

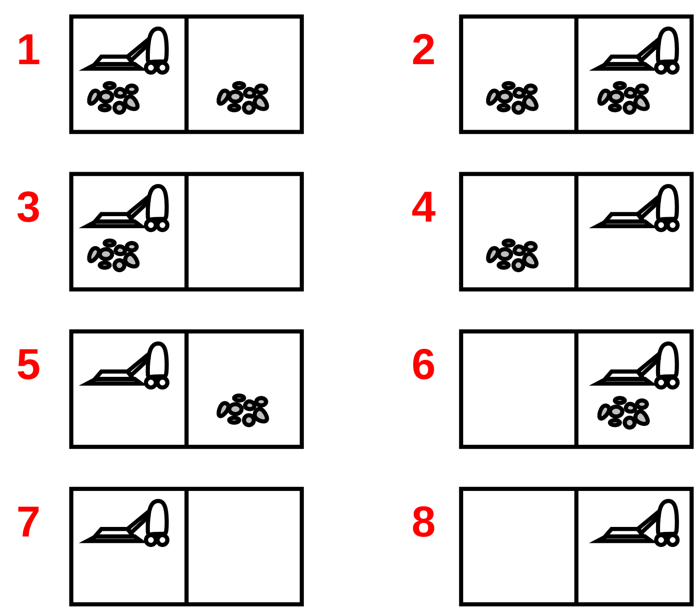
E.g., Vacuum World, no exogenous events and no sensors

Start in any of $\{1, 2, 3, 4, 5, 6, 7, 8\}$

Goal: have both rooms clean

Assume hitting the wall causes no harm
  $Left$ goes to $\{1, 3, 5, 7\}$
  $Right$ goes to $\{2, 4, 6, 8\}$

Solution: $[Right, Suck, Left, Suck]$



Problem types

Nondeterministic and/or **partially observable**

- percepts provide new information about current state
- solution is a *contingent plan* or a *policy*
- often **interleave** search, execution

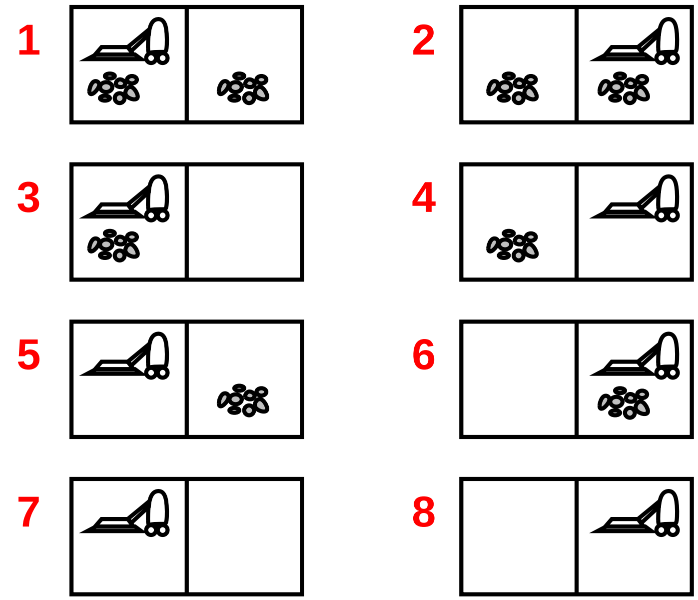
E.g., Vacuum World, no exogenous events, and local sensing:

which room the agent's in
and whether that room is dirty

Start in any of $\{5, 6, 7, 8\}$

Goal: have both rooms clean

Solution: $[Right, \mathbf{if}\ dirt\ \mathbf{then}\ Suck]$



Unknown state space $\Longrightarrow$ *exploration problem* (don't have example)

Problem-solving agents

Online problem solving: gather knowledge as you go
  Necessary for exploration problems
  Can be useful in nondeterministic and partially observable problems

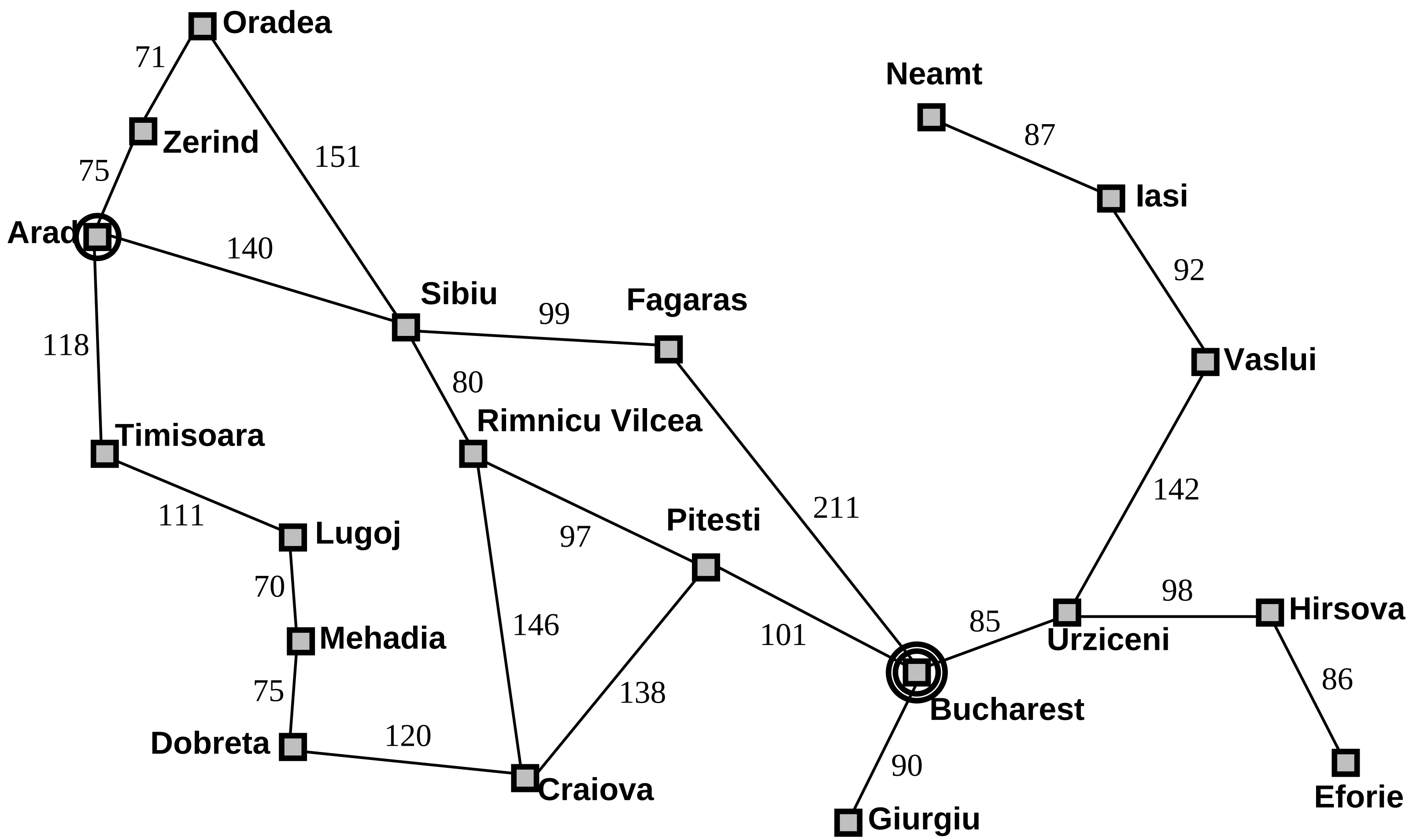
Offline problem solving: develop the entire solution at the start, before you ever start to execute it
  e.g., the solutions for the Vacuum World examples on the last three slides

Focus of this chapter: *offline* problem solving for *classical search problems* (i.e., deterministic, fully observable)

Example: Romania

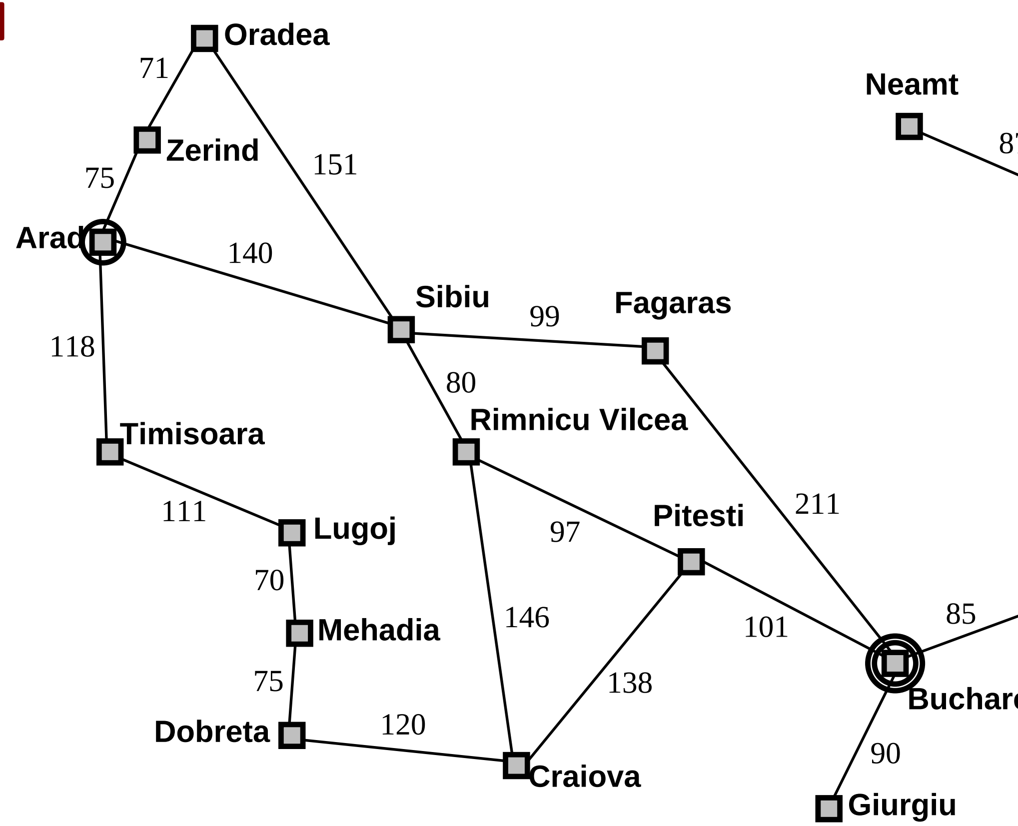
Currently in Arad, Romania; flight leaves tomorrow from Bucharest

states = cities; actions = drive between cities; goal = be in Bucharest



Selecting a state space

Real world is absurdly complex
⇒ state space must be **abstracted**
for problem solving



♢ *Abstract state* = set of real states
e.g., the state in-Arad
includes lots of locations

♢ *Abstract action* = complex
combination of real actions

e.g., goto-Zerind may include
possible routes, detours, rest stops,
etc. For guaranteed realizability,
it must get you to Zerind
no matter where you are in Arad

♢ *Abstract solution* = sequence of abstract actions
It represents a set of real paths that are solutions in the real world

Formulation of classical search problems

A *problem* consists of:

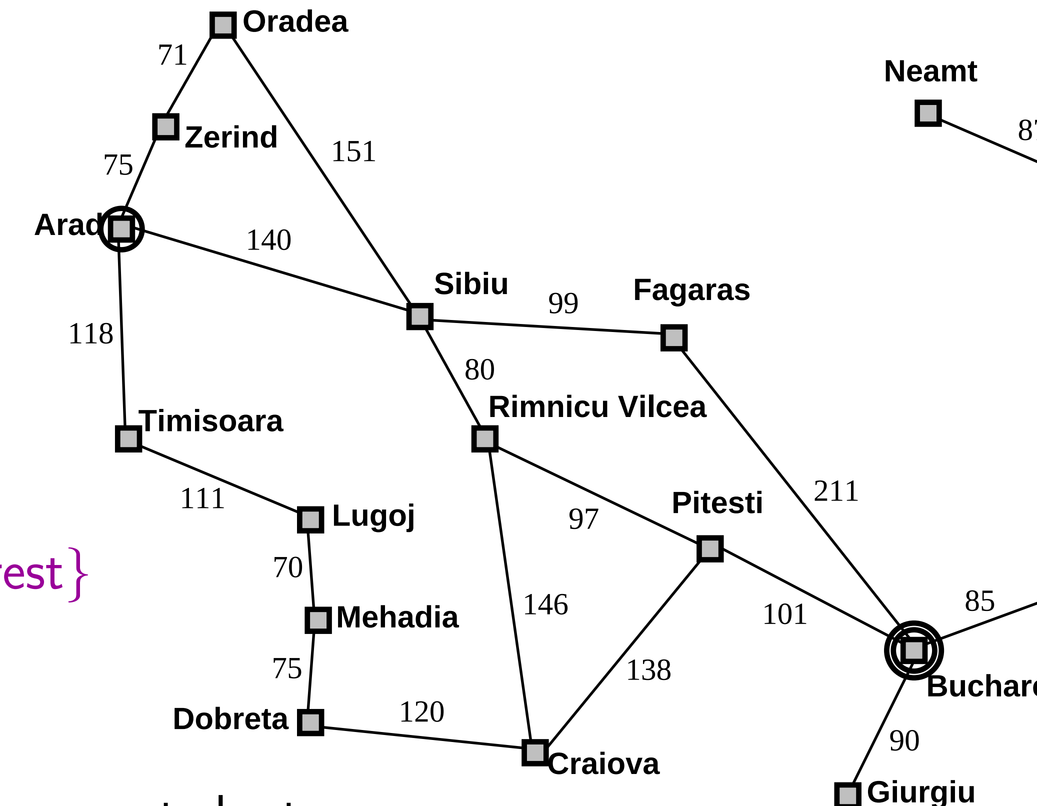
◇ *initial state* s_0, e.g., at-Arad

◇ *set of actions*, e.g.,
$A = \{\text{goto-Zerind}, \ldots\}$

◇ *state-transition function* $\gamma(s, a)$,
e.g., $\gamma(\text{at-Arad, goto-Zerind})$
$= \text{at-Zerind}, \ldots$

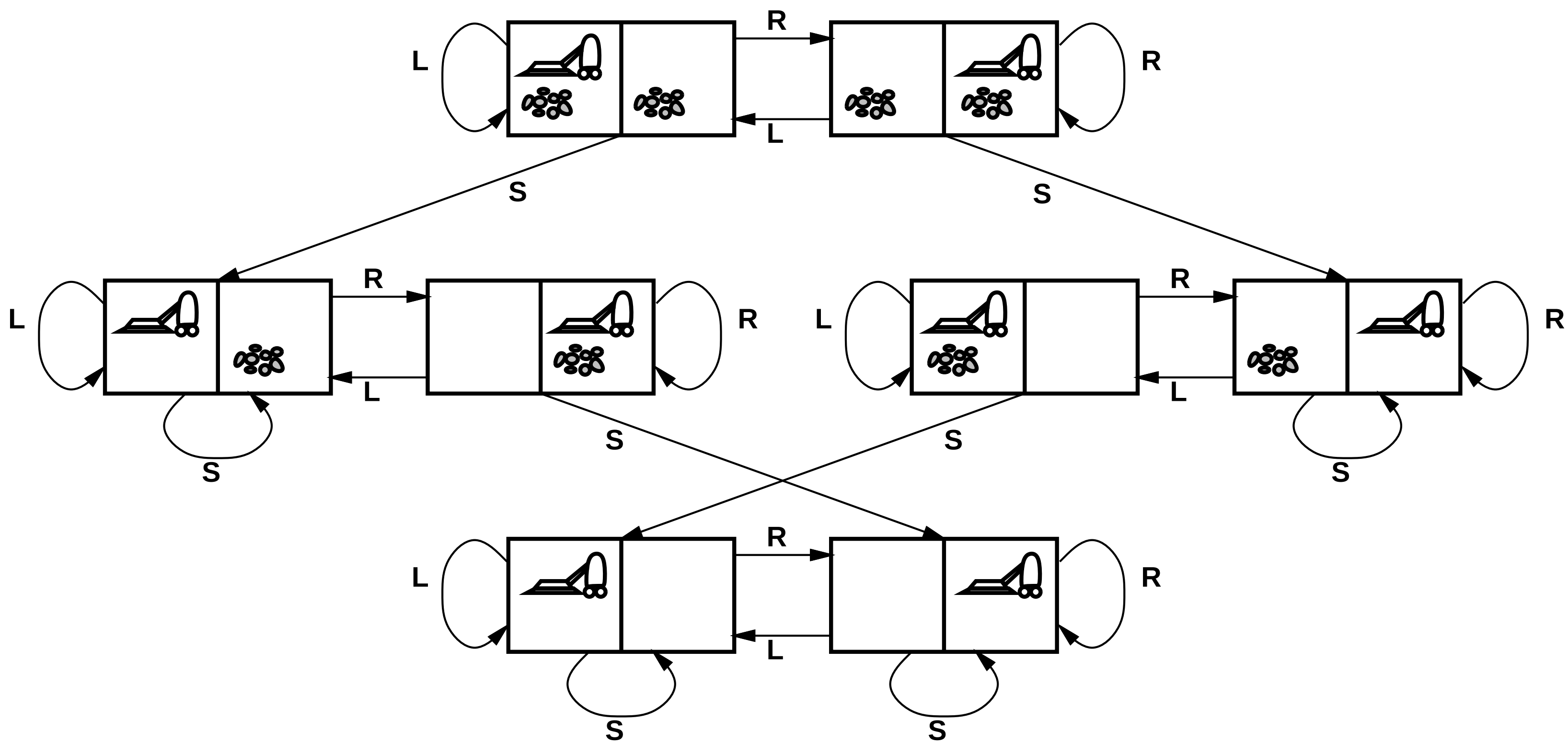
◇ *goal test* can be explicit, e.g.,
set of goal states $= \{\text{at-Bucharest}\}$
or implicit, e.g., $NoDirt(s)$

◇ *path cost* (additive), e.g.,
sum of distances, number of actions executed, etc.
$c(s, a)$ is the *step cost*, assumed to be ≥ 0



solution: sequence of actions leading from the initial state to a goal state

Example: vacuum world, no exogenous events



states: dirt and robot locations (ignore dirt amounts etc.)
actions: $Left$, $Right$, $Suck$, $NoOp$
goal test: no dirt
path cost: 1 per action (0 for $NoOp$)

Example: sliding-tile puzzles

$n \times n$ frame, $n^2 - 1$ movable tiles. Slide the tiles to change their positions.

$n = 3$: the 8-puzzle

7	2	4
5		6
8	3	1

a starting state

1	2	3
4	5	6
7	8	

goal state

$n = 4$: the 15-puzzle

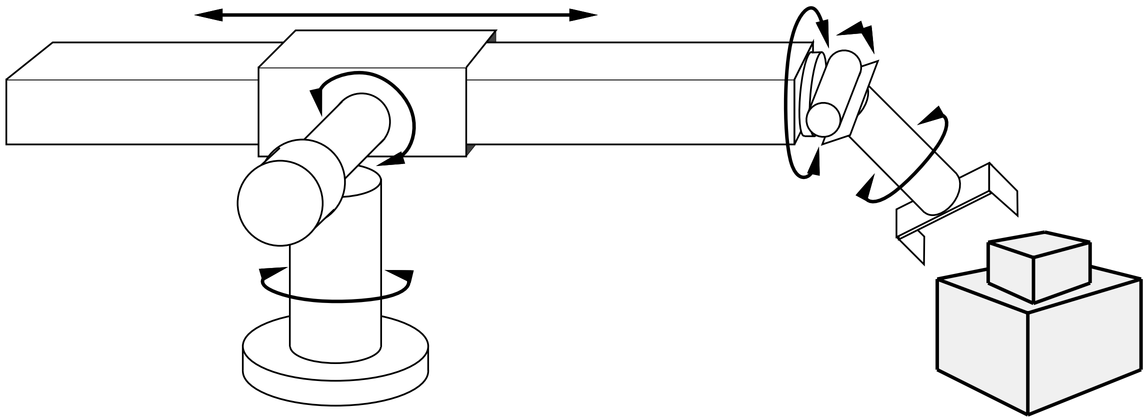
a starting state

goal state

- ♢ *states*: integer locations of tiles (ignore intermediate positions)
- ♢ *actions*: move tiles left, right, up, down (ignore unjamming etc.)
- ♢ *goal test* = goal state (shown)
- ♢ *step cost* = 1 per move, so *path cost* = number of moves

In this family of puzzles, finding optimal solutions is NP-hard
Easier if we don't care whether the solution is optimal

Example: robotic assembly



states: real-valued coordinates of robot joint angles
parts of the object to be assembled

actions: continuous motions of robot joints

goal test: complete assembly

path cost: time to execute

Tree search algorithms

Basic idea:

offline, simulated exploration of state space

```
function TREE-SEARCH( problem, strategy) returns a solution, or failure
    initialize the search tree using the initial state of problem
    loop do
        if there are no candidates for expansion then return failure
        choose a leaf node for expansion according to strategy
        if the node contains a goal state then return the corresponding solution
        else expand the node and add the resulting nodes to the search tree
    end
```

node: includes *state* s, *parent*, *children*, *depth*, *path cost* $g(s)$

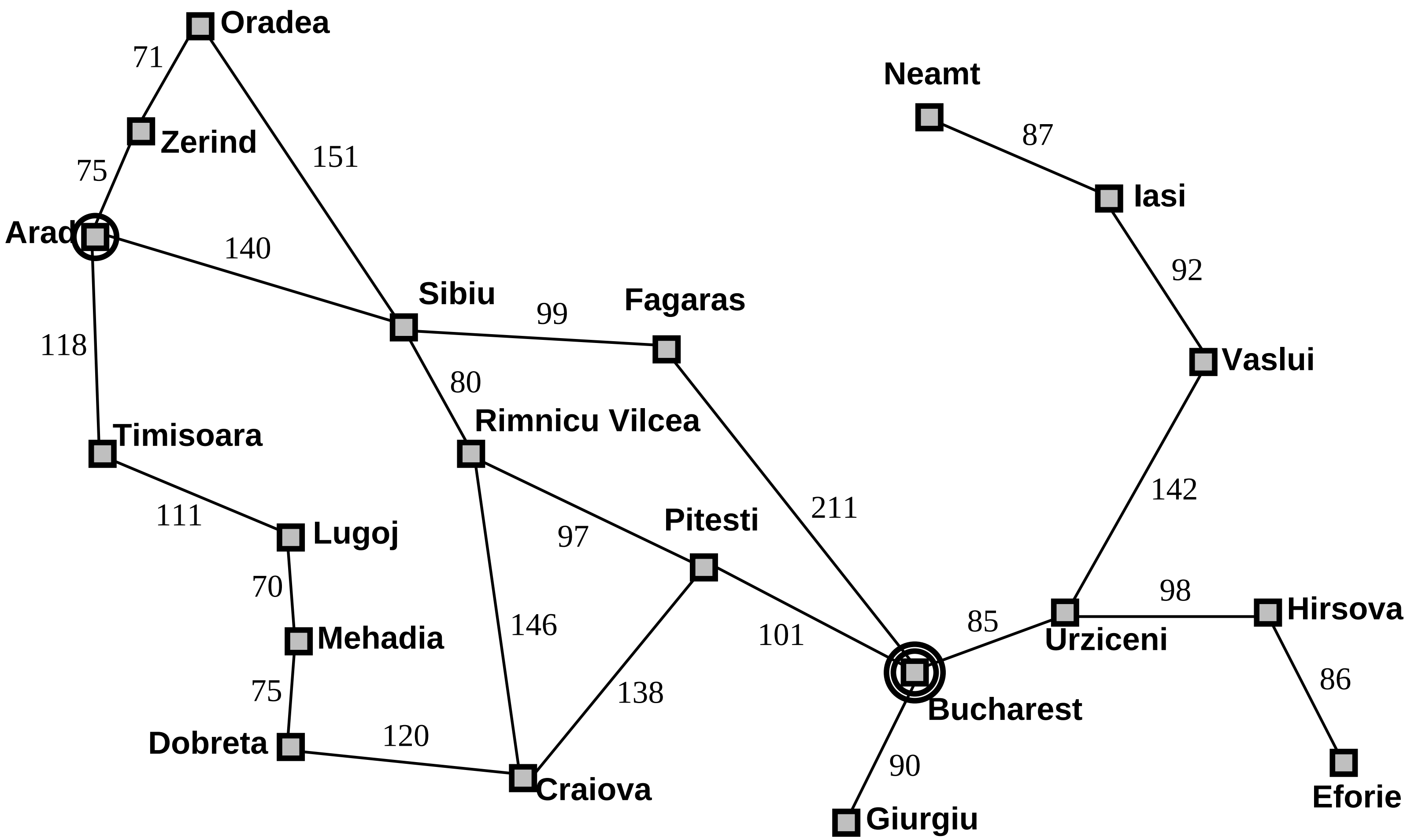
expanding a node: generating all of its children

fringe or *frontier* $= \{$all candidates for expansion$\}$

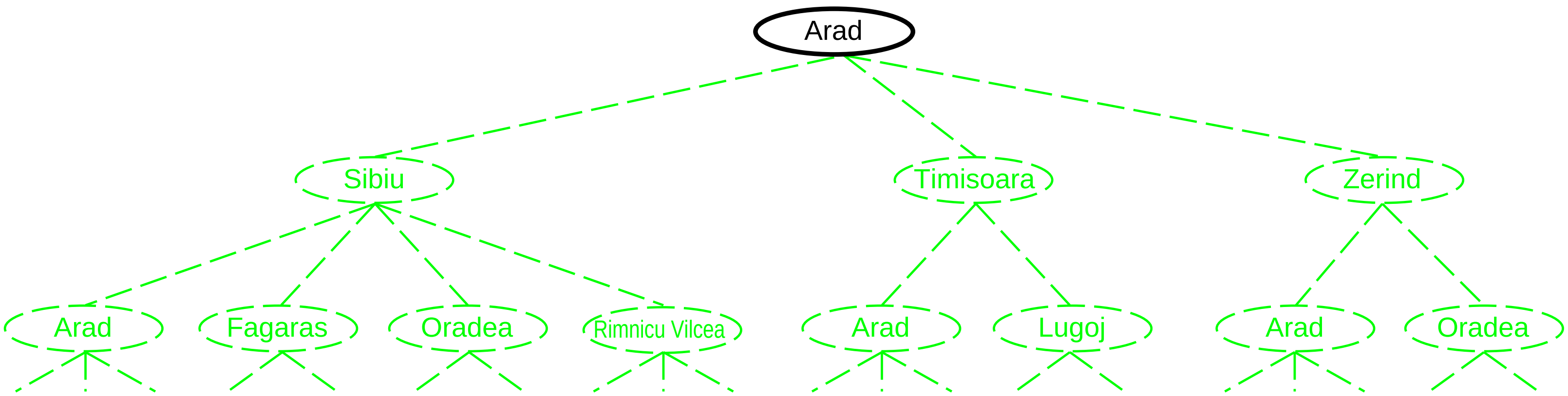
$= \{$all nodes that have been generated but not expanded$\}$

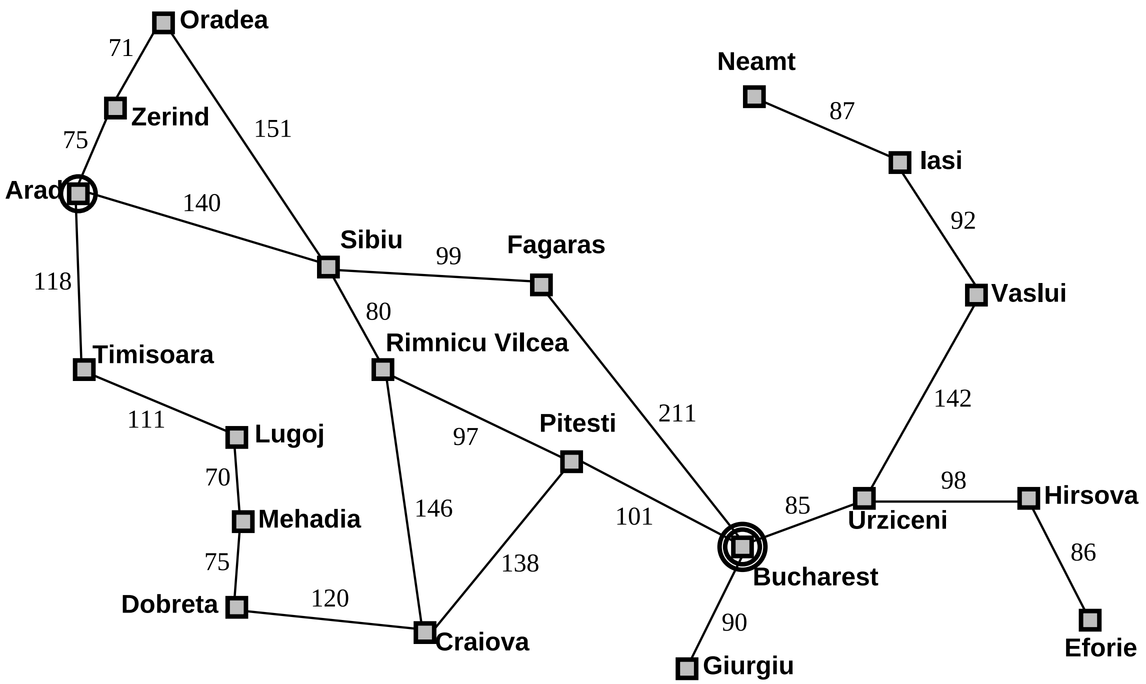
Tree search example

Currently in Arad, Romania; flight leaves tomorrow from Bucharest
states = cities; actions = drive between cities; goal = be in Bucharest

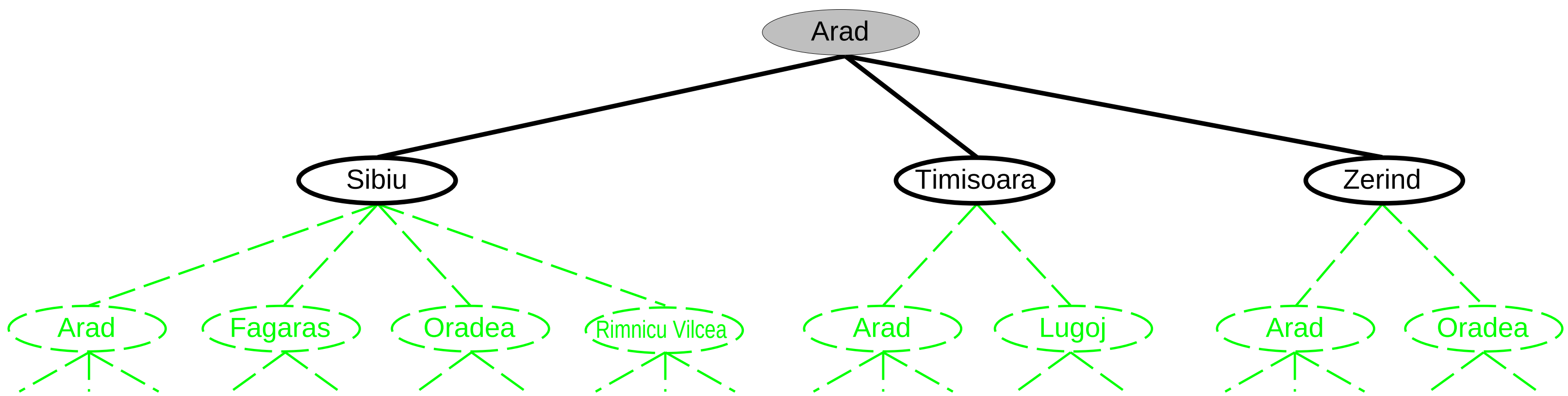


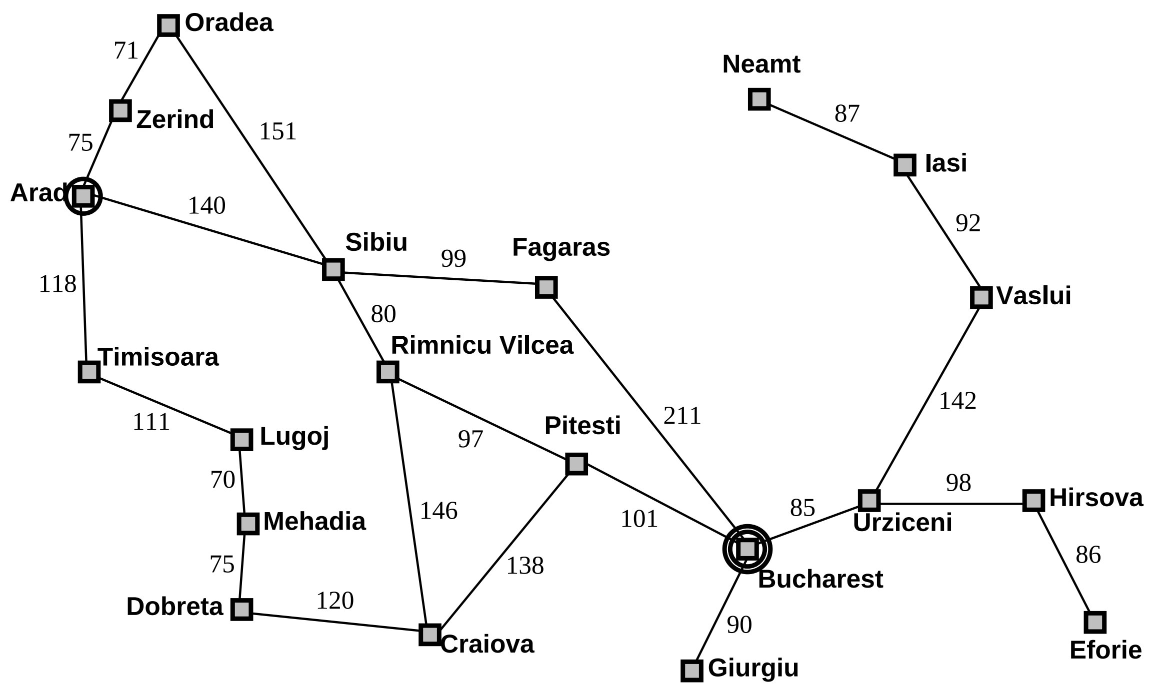
Tree search example



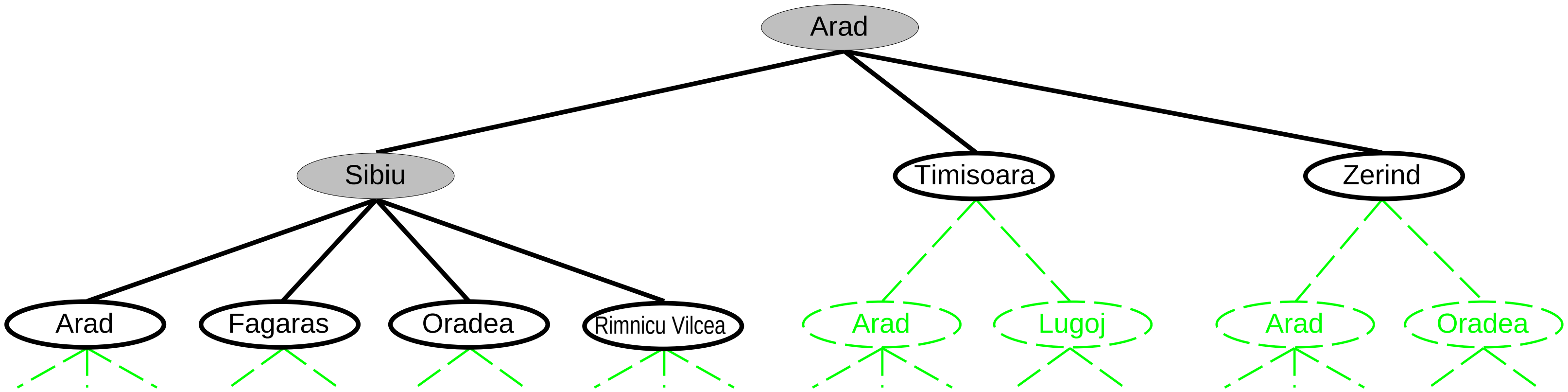


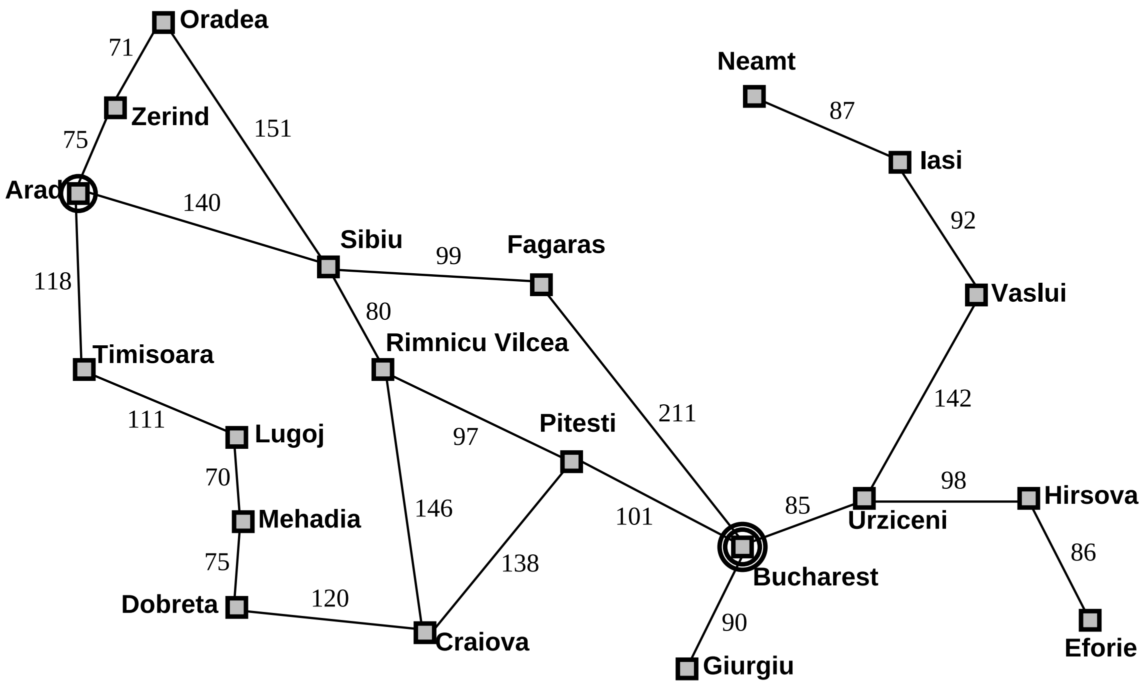
Tree search example





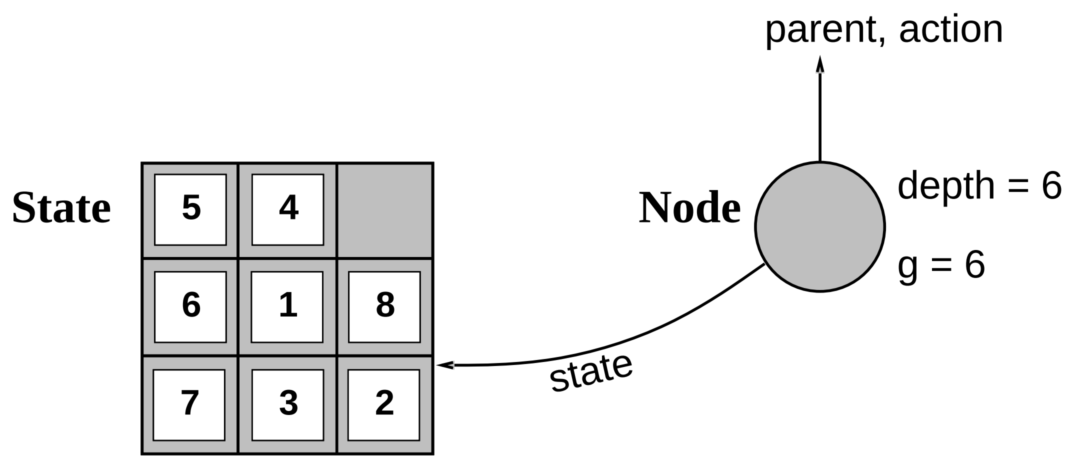
Tree search example





Implementation: states vs. nodes

- ♢ A *state* is a (representation of) a physical configuration
- ♢ A *node* x is a data structure that's part of a search tree. It includes state s, parent, children (if s has been expanded), depth, path cost $g(x)$
- ♢ The states themselves don't have parents, children, depth, or path cost



- ♢ The EXPAND function creates new nodes:
  - uses the state-transition function γ to generate the states for x's children: $\{\gamma(s, a) : a \text{ is applicable to } s\}$
  - fills in the various fields

Search strategies

A strategy is defined by picking the **order of node expansion**

Ways to evaluate a strategy:

completeness: does it always find a solution if one exists?
optimality: does it always find a least-cost solution?
time complexity: number of nodes generated/expanded
space complexity: maximum number of nodes in memory

Time and space complexity are measured in terms of

$b =$ maximum branching factor of the search tree; we'll assume it's finite
$d =$ depth of the least-cost solution (or ∞ if there's no solution)
$m =$ maximum depth of the state space (may be ∞)

Uninformed search strategies

Uninformed strategies use only the information available in the problem definition

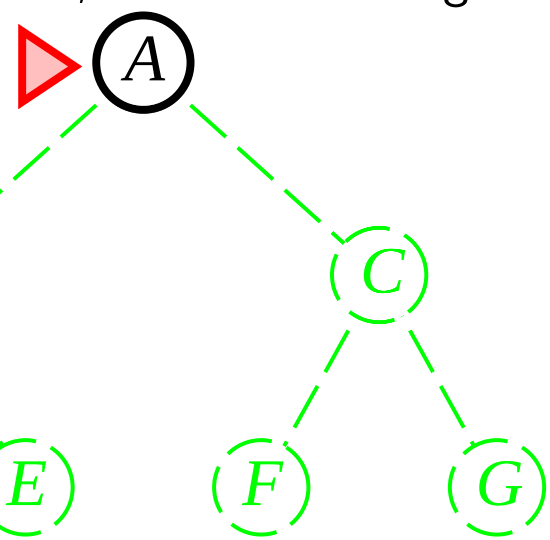
- ♢ Breadth-first search
- ♢ Depth-first search
- ♢ Depth-limited search
- ♢ Uniform-cost search
- ♢ Iterative deepening search

Breadth-first search

Expand shallowest unexpanded node

Implementation:

$fringe$ is a FIFO queue, i.e., new successors go at end

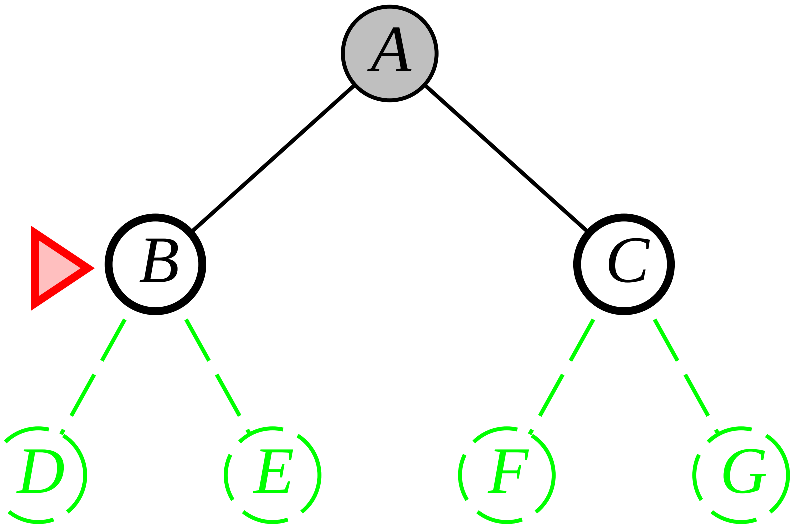


Breadth-first search

Expand shallowest unexpanded node

Implementation:

fringe is a FIFO queue, i.e., new successors go at end

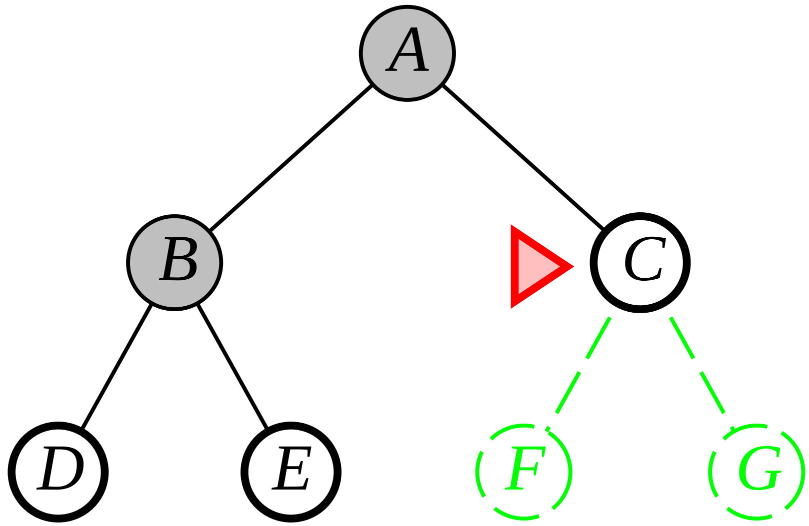


Breadth-first search

Expand shallowest unexpanded node

Implementation:

$fringe$ is a FIFO queue, i.e., new successors go at end

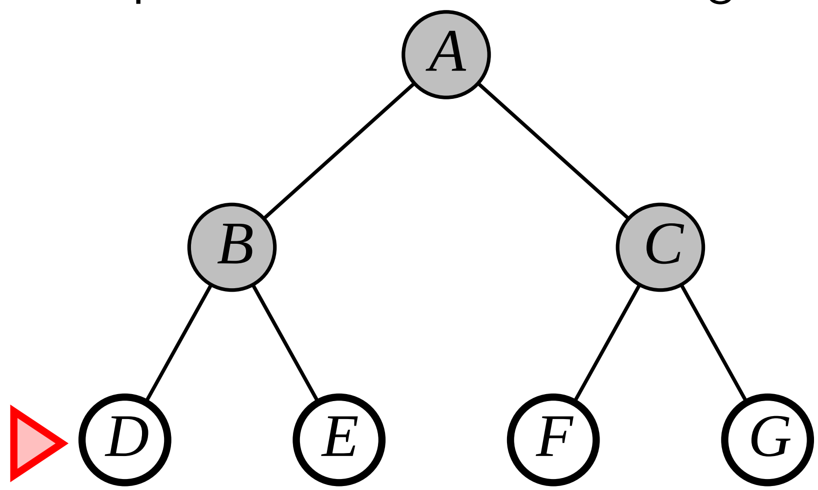


Breadth-first search

Expand shallowest unexpanded node

Implementation:

fringe is a FIFO queue, i.e., new successors go at end



Properties of breadth-first search

Complete??

$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Properties of breadth-first search

Complete? Yes

Time?

$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Properties of breadth-first search

Complete? Yes

Time? $1 + b + b^2 + b^3 + \ldots + b^d + b(b^d - 1) = O(b^d)$, i.e., exp. in d

Space?

$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Properties of breadth-first search

Complete? Yes

Time? $1 + b + b^2 + b^3 + \ldots + b^d + b(b^d - 1) = O(b^d)$, i.e., exp. in d

Space? $O(b^d)$ (keeps every node in memory)

This is a big problem. If we run for 24 hours and generate nodes at 100MB/sec, the space requirement is 8.64 TB

Optimal solutions?

$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Properties of breadth-first search

Complete? Yes

Time? $1 + b + b^2 + b^3 + \ldots + b^d + b(b^d - 1) = O(b^d)$, i.e., exp. in d

Space? $O(b^d)$ (keeps every node in memory)

This is a big problem. If we run for 24 hours and generate nodes at 100MB/sec, the space requirement is 8.64 TB

Optimal solutions? Yes if cost $=$ 1 per step, but not in general

---

$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Uniform-cost search

Expand least-cost unexpanded node

Implementation: $fringe =$ queue ordered by path cost, lowest first

Equivalent to breadth-first if step costs all equal

Complete?

$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Uniform-cost search

Expand least-cost unexpanded node

Implementation: fringe = queue ordered by path cost, lowest first

Equivalent to breadth-first if step costs all equal

Complete? Yes, if $\exists\ \epsilon > 0$ such that step cost $\geq \epsilon$

Time?

$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Uniform-cost search

Expand least-cost unexpanded node

Implementation: $fringe =$ queue ordered by path cost, lowest first

Equivalent to breadth-first if step costs all equal

Complete? Yes, if $\exists\ \epsilon > 0$ such that step cost $\geq \epsilon$

Time? $\#$ of nodes with $g \leq$ cost of optimal solution, $O(b^{\lceil C^*/\epsilon \rceil})$
where C^* is the cost of the optimal solution

Space?

$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Uniform-cost search

Expand least-cost unexpanded node

Implementation: $fringe =$ queue ordered by path cost, lowest first

Equivalent to breadth-first if step costs all equal

Complete? Yes, if $\exists\ \epsilon > 0$ such that step cost $\geq \epsilon$

Time? # of nodes with $g \leq$ cost of optimal solution, $O(b^{\lceil C^*/\epsilon \rceil})$
where C^* is the cost of the optimal solution

Space? # of nodes with $g \leq$ cost of optimal solution, $O(b^{\lceil C^*/\epsilon \rceil})$

Optimal solutions?

$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Uniform-cost search

Expand least-cost unexpanded node

Implementation: $fringe =$ queue ordered by path cost, lowest first

Equivalent to breadth-first if step costs all equal

Complete? Yes, if $\exists\ \epsilon > 0$ such that step cost $\geq \epsilon$

Time? # of nodes with $g \leq$ cost of optimal solution, $O(b^{\lceil C^*/\epsilon \rceil})$
where C^* is the cost of the optimal solution

Space? # of nodes with $g \leq$ cost of optimal solution, $O(b^{\lceil C^*/\epsilon \rceil})$

Optimal solutions? Yes

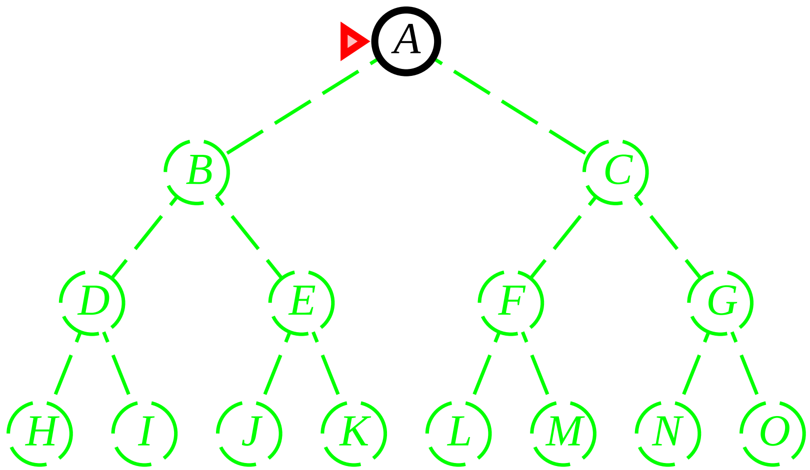
$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Depth-first search

Expand deepest unexpanded node

Implementation:

$fringe$ = LIFO queue, i.e., put successors at front

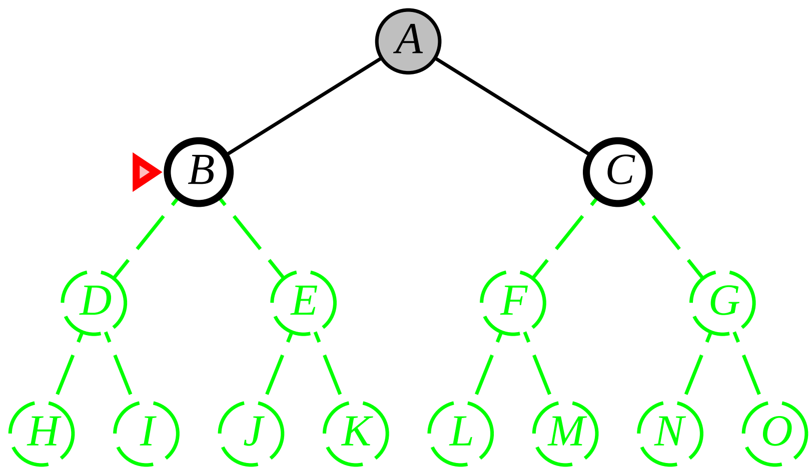


Depth-first search

Expand deepest unexpanded node

Implementation:

$fringe$ = LIFO queue, i.e., put successors at front

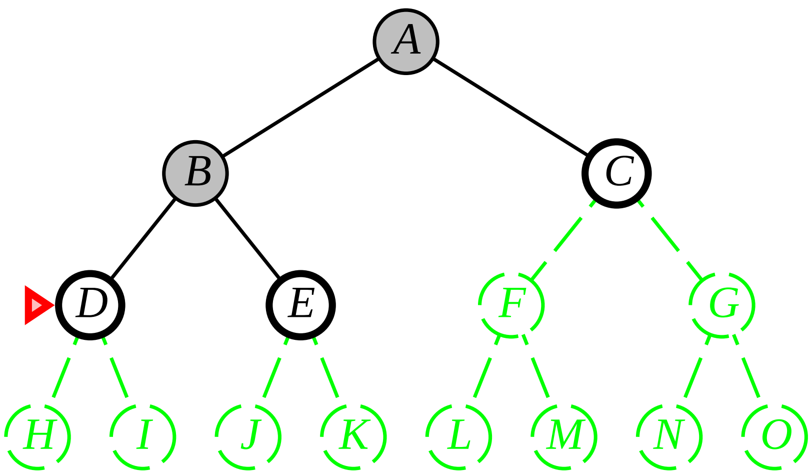


Depth-first search

Expand deepest unexpanded node

Implementation:

$fringe$ = LIFO queue, i.e., put successors at front

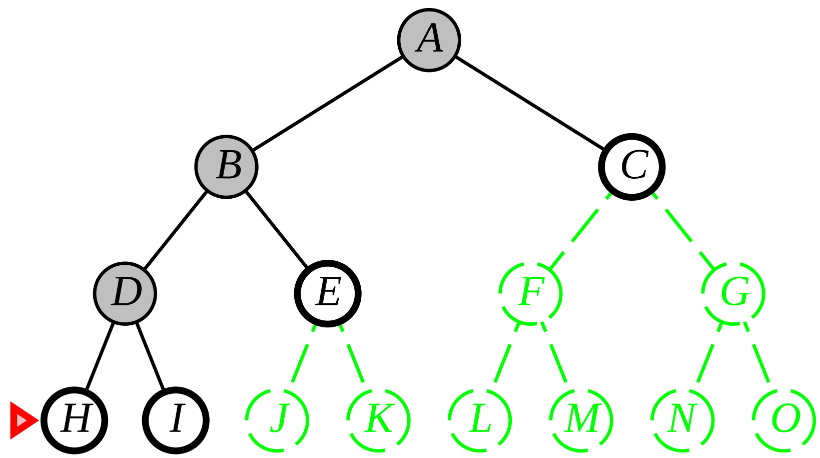


Depth-first search

Expand deepest unexpanded node

Implementation:

$fringe$ = LIFO queue, i.e., put successors at front

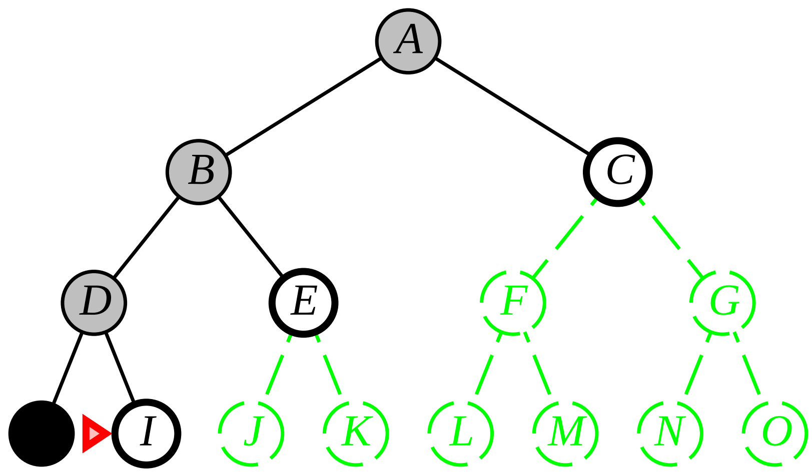


Depth-first search

Expand deepest unexpanded node

Implementation:

$fringe$ = LIFO queue, i.e., put successors at front

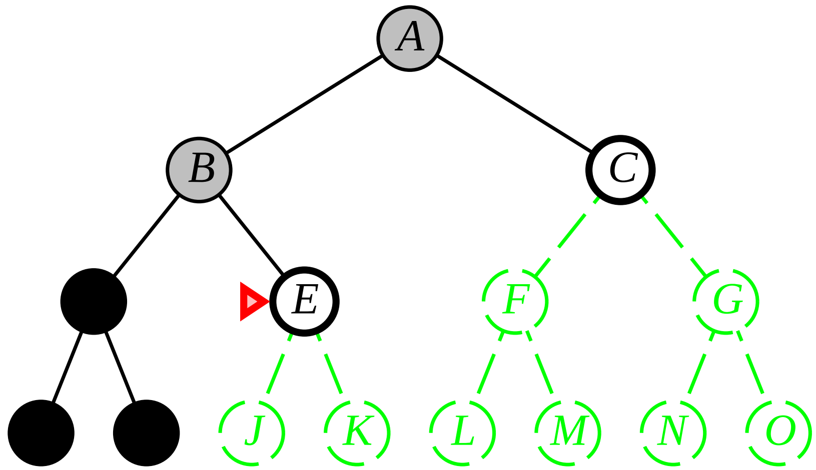


Depth-first search

Expand deepest unexpanded node

Implementation:

$fringe$ = LIFO queue, i.e., put successors at front

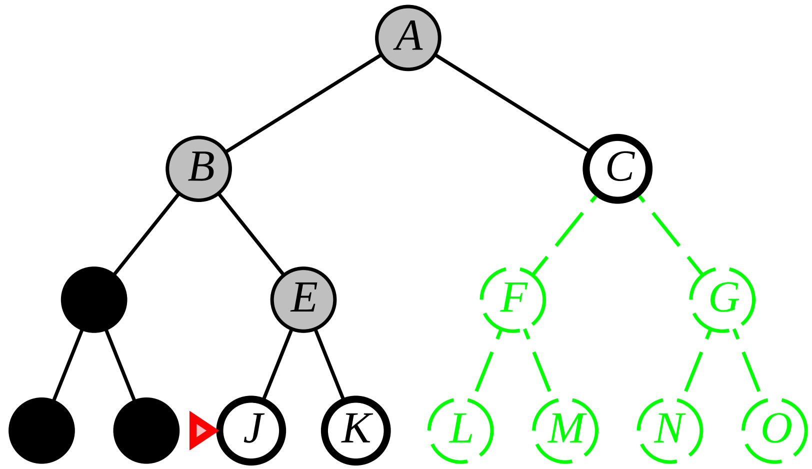


Depth-first search

Expand deepest unexpanded node

Implementation:

$fringe$ = LIFO queue, i.e., put successors at front

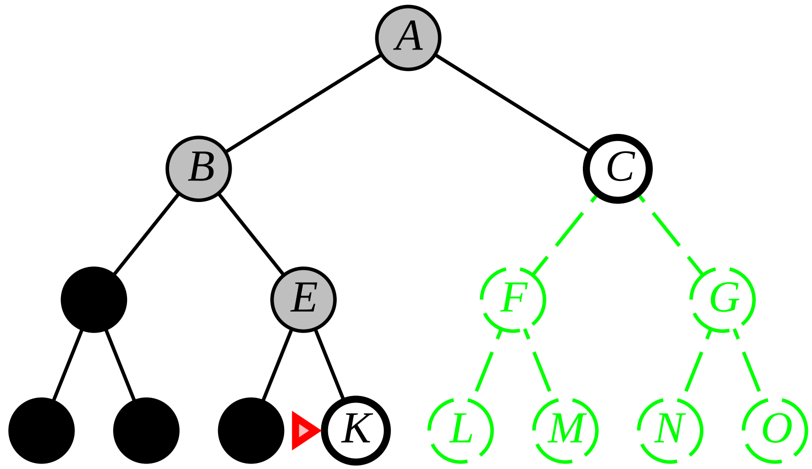


Depth-first search

Expand deepest unexpanded node

Implementation:

$fringe$ = LIFO queue, i.e., put successors at front

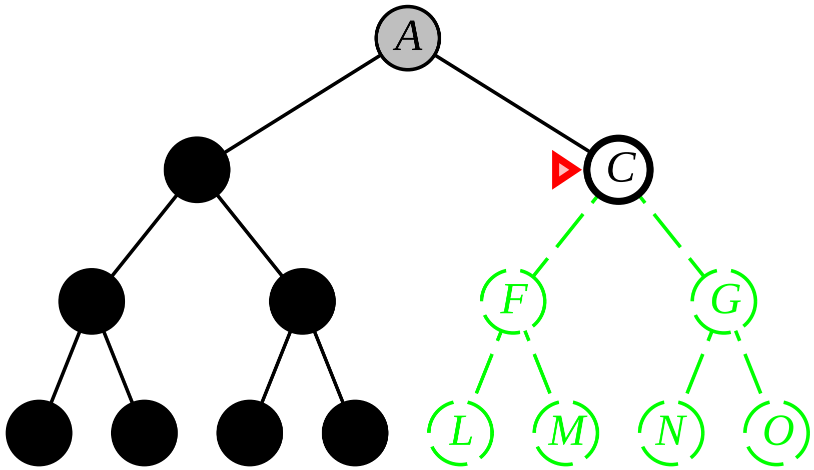


Depth-first search

Expand deepest unexpanded node

Implementation:

$fringe$ = LIFO queue, i.e., put successors at front

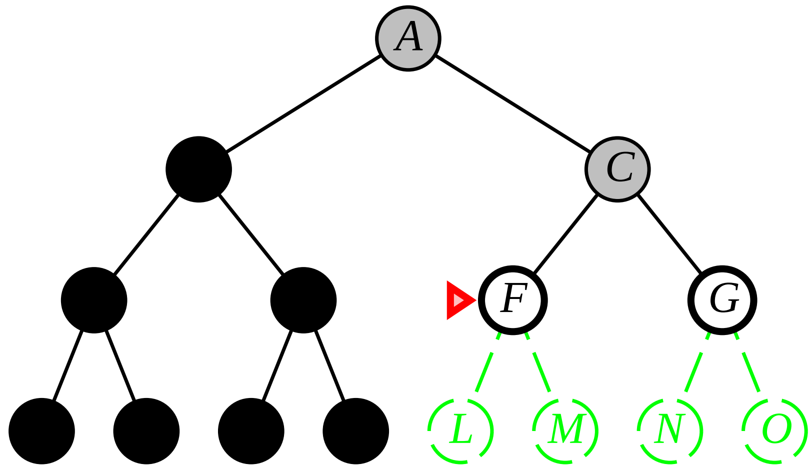


Depth-first search

Expand deepest unexpanded node

Implementation:

$fringe$ = LIFO queue, i.e., put successors at front

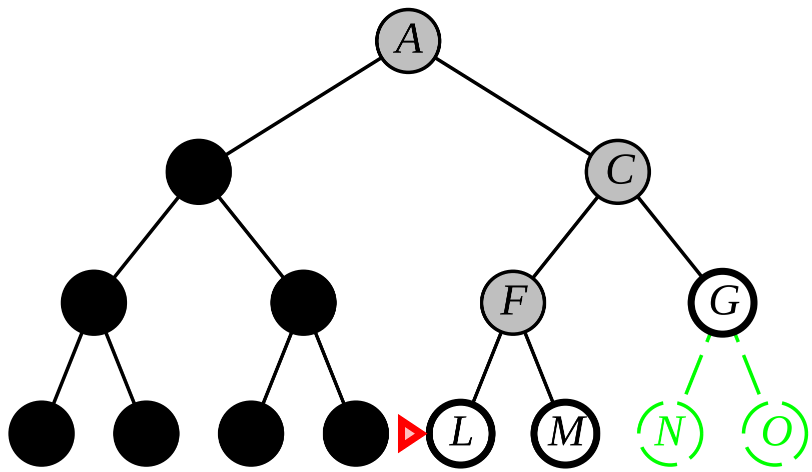


Depth-first search

Expand deepest unexpanded node

Implementation:

$fringe =$ LIFO queue, i.e., put successors at front

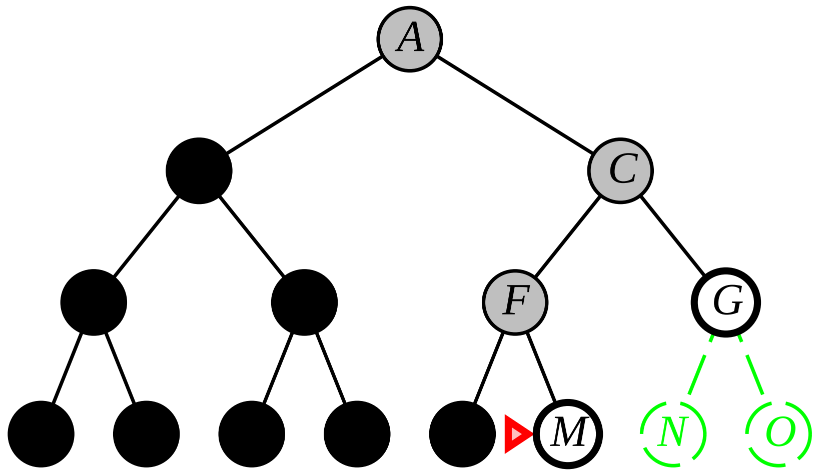


Depth-first search

Expand deepest unexpanded node

Implementation:

$fringe$ = LIFO queue, i.e., put successors at front



Properties of depth-first search

Complete??

$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Properties of depth-first search

Complete?

No in infinite-depth spaces

Yes in finite spaces, if we modify to avoid loops:

Backtrack if you reach a state you've already seen on the current path

Time?

$b =$ maximum branching factor of the search tree

$d =$ depth of the least-cost solution

$m =$ maximum depth of the state space (may be ∞)

Properties of depth-first search

Complete?

No in infinite-depth spaces

Yes in finite spaces, if we modify to avoid loops:

Backtrack if you reach a state you've already seen on the current path

Time? $O(b^m)$: terrible if m is much larger than d

but if solutions are dense, may be much faster than breadth-first

Space?

$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Properties of depth-first search

Complete?

No in infinite-depth spaces

Yes in finite spaces, if we modify to avoid loops:

Backtrack if you reach a state you've already seen on the current path

Time? $O(b^m)$: terrible if m is much larger than d

but if solutions are dense, may be much faster than breadth-first

Space? $O(bm)$, i.e., linear space

Optimal solutions?

$b =$ maximum branching factor of the search tree

$d =$ depth of the least-cost solution

$m =$ maximum depth of the state space (may be ∞)

Properties of depth-first search

Complete?
No in infinite-depth spaces
Yes in finite spaces, if we modify to avoid loops:
Backtrack if you reach a state you've already seen on the current path

Time? $O(b^m)$: terrible if m is much larger than d
but if solutions are dense, may be much faster than breadth-first

Space? $O(bm)$, i.e., linear space

Optimal solutions? Not unless it's lucky

$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Depth-limited search

Depth-first search, backtrack at each node of depth l unless it's a solution

Recursive implementation:

```
function DEPTH-LIMITED-SEARCH( problem, limit) returns soln/fail/cutoff
   RECURSIVE-DLS(MAKE-NODE(INITIAL-STATE[problem]), problem, limit)

function RECURSIVE-DLS(node, problem, limit) returns soln/fail/cutoff
   cutoff-occurred? ← false
   if GOAL-TEST(problem, STATE[node]) then return node
   else if DEPTH[node] = limit then return cutoff
   else for each successor in EXPAND(node, problem) do
      result ← RECURSIVE-DLS(successor, problem, limit)
      if result = cutoff then cutoff-occurred? ← true
                                 /* tells what to return if we don't find a solution */
      else if result ≠ failure then return result
   if cutoff-occurred? then return cutoff else return failure
```

Iterative deepening search

Depth-limited search to depth 0,
Depth-limited search to depth 1,
Depth-limited search to depth 2,
...
Stop when you find a solution

```
function ITERATIVE-DEEPENING-SEARCH(problem) returns a solution
    inputs: problem, a problem

    for depth ← 0 to ∞ do
        result ← DEPTH-LIMITED-SEARCH(problem, depth)
        if result ≠ cutoff then return result
    end
```

Iterative deepening search

Limit = 0

function Iterative-Deepening-Search(*problem*) **returns** a solution
  inputs: *problem*, a problem

  for *depth* ← 0 **to** ∞ **do**
    result ← Depth-Limited-Search(*problem*, *depth*)
    if *result* $\neq$ cutoff **then return** *result*
  end

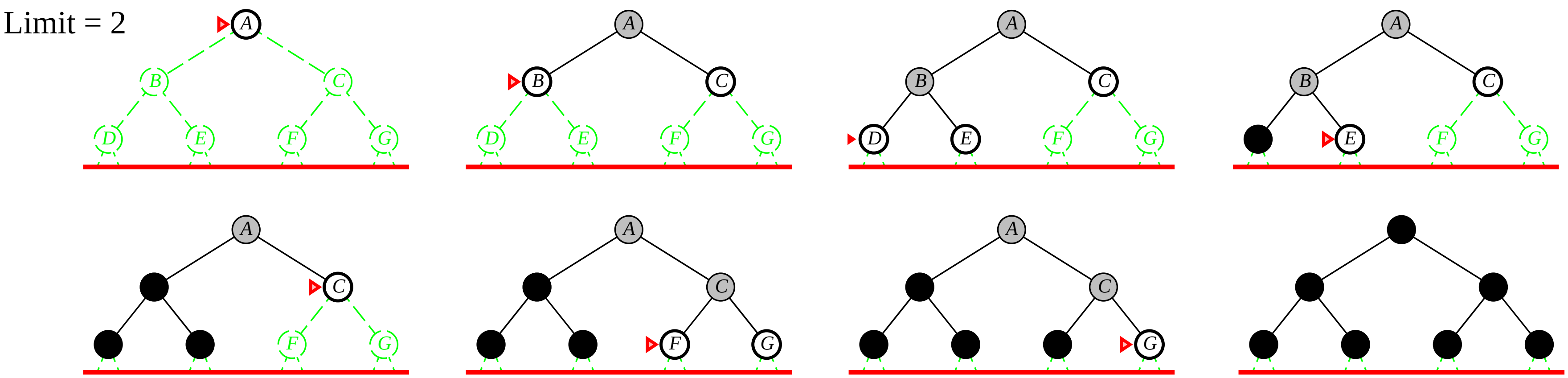
Iterative deepening search

Limit = 1

function ITERATIVE-DEEPENING-SEARCH(*problem*) **returns** a solution
    inputs: *problem*, a problem

    for *depth* $\leftarrow$ 0 **to** ∞ **do**
        result $\leftarrow$ DEPTH-LIMITED-SEARCH(*problem, depth*)
        if *result* $\neq$ cutoff **then return** *result*
    end

Iterative deepening search

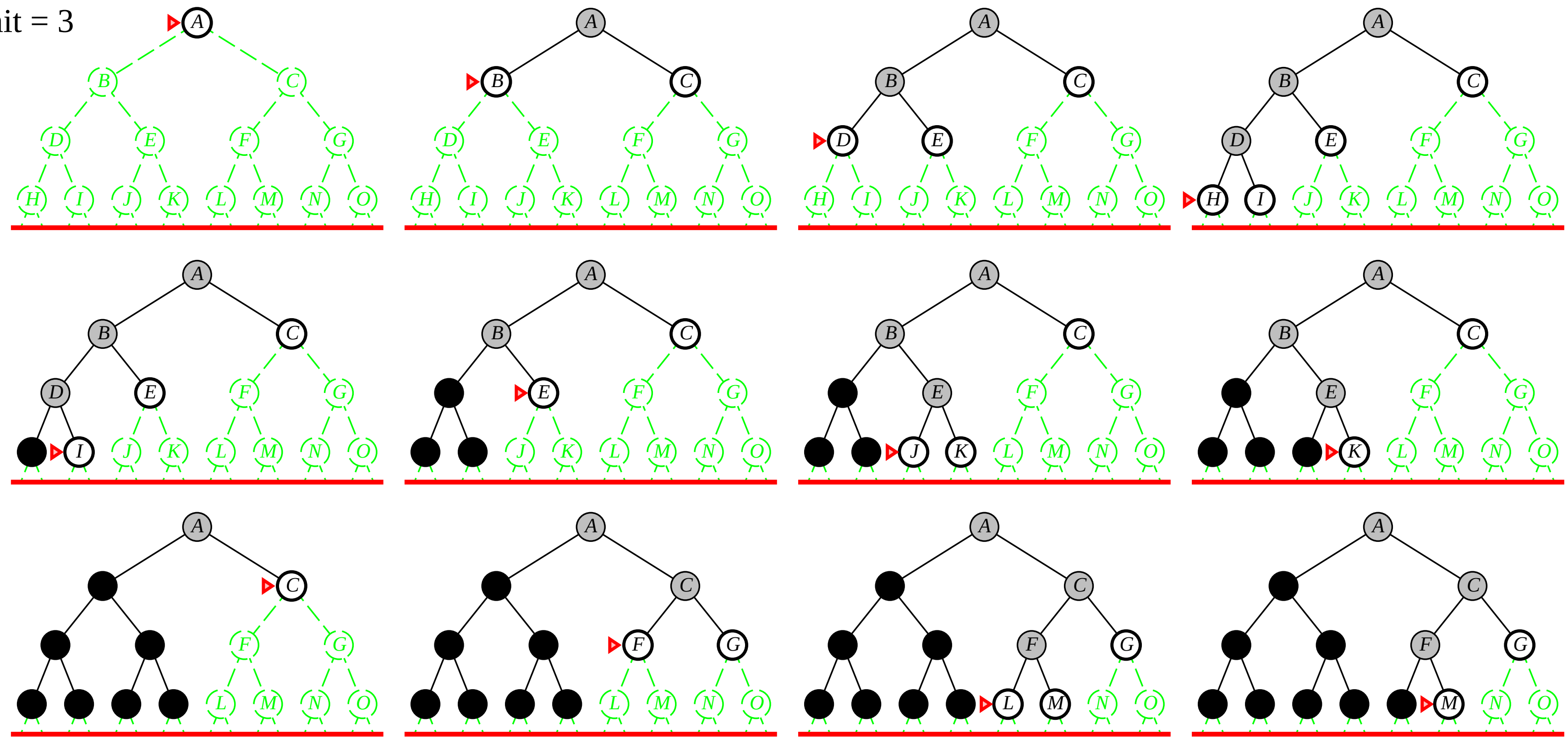


function ITERATIVE-DEEPENING-SEARCH(*problem*) **returns** a solution
  inputs: *problem*, a problem

  for *depth* ← 0 **to** ∞ **do**
    result ← DEPTH-LIMITED-SEARCH(*problem*, *depth*)
    if *result* $\neq$ cutoff **then return** *result*
  end

Iterative deepening search

Limit = 3



Properties of iterative deepening search

Complete??

$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Properties of iterative deepening search

Complete? Yes

Time?

$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Properties of iterative deepening search

Complete? Yes

Time? $(d+1)b^0 + db^1 + (d-1)b^2 + \ldots + b^d = O(b^d)$

Space?

$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Properties of iterative deepening search

Complete? Yes

Time? $(d+1)b^0 + db^1 + (d-1)b^2 + \ldots + b^d = O(b^d)$

Space? $O(bd)$

Optimal solutions?

$b =$ maximum branching factor of the search tree
$d =$ depth of the least-cost solution
$m =$ maximum depth of the state space (may be ∞)

Properties of iterative deepening search

Complete? Yes

Time? $(d+1)b^0 + db^1 + (d-1)b^2 + \ldots + b^d = O(b^d)$

Space? $O(bd)$

Optimal solutions? Yes, if step cost $= 1$
Can be modified to behave like uniform-cost search

Node-generation operations for $b = 10$ and $d = 5$, solution at far right leaf:

IDS: $1 + 50 + 400 + 3,000 + 20,000 + 100,000 = 123,450$
BFS: $1 + 10 + 100 + 1,000 + 10,000 + 100,000 + 999,990 = 1,111,100$

IDS does better because it doesn't expand the nodes at depth d

BFS expands them because of a quirk in the pseudocode

Tree search

```
function TREE-SEARCH( problem, strategy) returns a solution, or failure
    initialize the search tree using the initial state of problem
    loop do
        if there are no candidates for expansion then return failure
        choose a leaf node for expansion according to strategy
        if the node contains a goal state then return the corresponding solution
        else expand the node and add the resulting nodes to the search tree
    end
```

TREE-SEARCH doesn't do the goal test until it selects a node for expansion

- ◇ Needed for uniform-cost search to find optimal solutions
- ◇ Needed for some algorithms in the next chapter

With breadth-first search, we're looking for shallowest
(but not necessarily optimal) solutions

Modify the pseudocode to check for a solution whenever a node is generated

Tree search for BFS

function TREE-SEARCH(*problem, strategy*) **returns** a solution, or failure
  initialize the search tree using the initial state of *problem*
  loop do
    if there are no candidates for expansion **then return** failure
    choose a leaf node for expansion according to *strategy*
    if the node contains a goal state **then return** the corresponding solution
    else expand the node and add the resulting nodes to the search tree
  end

↑ **Modification:** *if any of them is a solution, return it immediately*

Number of node-generation operations:

IDS: $1 + 50 + 400 + 3,000 + 20,000 + 100,000 = 123,450$

BFS: $1 + 10 + 100 + 1,000 + 10,000 + 100,000 = 111,110$

Highest number of nodes stored:

IDS: $1 + 10 \times 5 = 51$

BFS: $1 + 10 + 100 + 1,000 + 10,000 + 100,000 = 111,111$

Summary of algorithms

Criterion	Breadth-First	Uniform-Cost	Depth-First	Depth-Limited	Iterative Deepening
Complete?	Yes	Yes$^{(2)}$	No	Yes, if $l \geq d$	Yes
Time	b^d	$b^{\lceil C^*/\epsilon \rceil}$	b^m	b^l	b^d
Space	b^d	$b^{\lceil C^*/\epsilon \rceil}$	bm	bl	bd
Optimal?	Yes$^{(1)}$	Yes	No	No	Yes$^{(1)}$

where

b = branching factor

C^* = cost of optimal solution, or ∞ if there's no solution

d = depth of shallowest solution, or ∞ if there's no solution

ϵ = smallest cost of each edge

l = cutoff depth for depth-limited search

m = depth of deepest node (may be ∞)

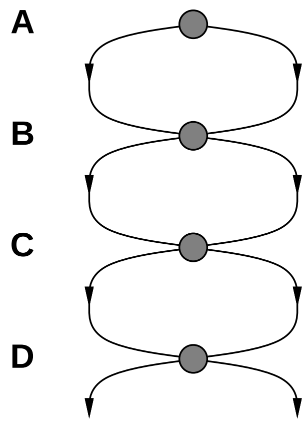
---

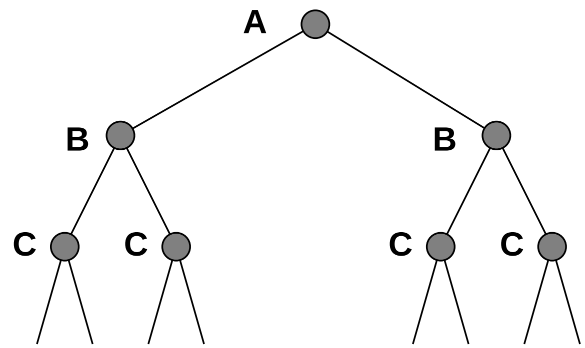
[1] if step cost is 1

[2] if $\epsilon > 0$

Repeated states

Failure to detect repeated states can turn a linear problem into an exponential one!





Graph search

```
function GRAPH-SEARCH( problem, fringe) returns a solution, or failure
    closed ← an empty set
    fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
    loop do
        if fringe is empty then return failure
        node ← REMOVE-FRONT(fringe)
        if GOAL-TEST(problem, STATE[node]) then return node
        if STATE[node] is not in closed then
            add STATE[node] to closed
            fringe ← INSERTALL(EXPAND(node, problem), fringe)
    end
```

Can do breadth-first graph search, uniform-cost graph search

Can also do depth-first graph search, but there's a tradeoff:

- ♢ Sometimes get exponentially less time than depth-first tree search
- ♢ Usually need exponentially more memory than depth-first tree search

Summary

◇ Problem formulation usually requires abstracting away real-world details to define a state space that can feasibly be explored

◇ Variety of uninformed search strategies

◇ Iterative deepening search uses only linear space
and (when $b \geq 2$) not much more time than other uninformed algorithms

◇ Graph search sometimes takes exponentially less time than tree search (when the number of paths to a node is exponential in its depth)

◇ Graph search sometimes takes exponentially more space than tree search (when the search space is treelike)

Homework assignment (due in one week)
five problems, 10 points each – total 50 points

2.9, 3.7(a,b), 3.8, 3.9(a,c), 3.13