MAPL 600 / CMSC 760 Fall 2007
Take-Home Exam 3
Show all work.
All work must be your own (i.e., no group efforts are allowed).
If you use a reference, cite it, or you will lose credit!
Work problems totaling 50 points.
(I'll stop grading after that, so don't hand in extra parts.)
Due: Friday Nov 2, 8am. (See late penalty policy on information sheet.)

NOTATION: In all of these problems, assume that $\mathbf{A}$ and $\mathbf{M}$ are symmetric and positive definite, with $\mathbf{A} = \mathbf{M} - \mathbf{N}$.

1a. (10) Show that if the rank of the $n \times n$ matrix $\mathbf{R}$ is $k < n$, then the matrix $\mathbf{I} + \mathbf{R}$ has $n - k$ eigenvalues equal to 1.

1b. (10) Suppose the rank of $\mathbf{N}$ is $k < n$. How many iterations will cg take to solve the linear system $\mathbf{Ax} = \mathbf{b}$ using $\mathbf{M}$ as a preconditioner? Justify your answer.

2a. (10) Prove that

$$(\mathbf{A} - \mathbf{Z}\mathbf{V}^T)^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{Z}(\mathbf{I} - \mathbf{V}^T\mathbf{A}^{-1}\mathbf{Z})^{-1}\mathbf{V}^T\mathbf{A}^{-1}$$

by verifying that the product of this matrix with $\mathbf{A} - \mathbf{Z}\mathbf{V}^T$ is the identity matrix $\mathbf{I}$. Dimensions: $\mathbf{A}$ is $n \times n$, and $\mathbf{Z}$ and $\mathbf{V}$ are $n \times k$ with $k < n$. (In particular, $\mathbf{Z}$ and $\mathbf{V}$ are not square so they have no inverse.) This is called the Sherman-Morrison-Woodbury (SMW) Formula.

2b. (10) Write an algorithm that uses the SMW formula and a factorization $\mathbf{M} = \mathbf{LU}$ to solve the linear system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{N} = \mathbf{E}\mathbf{E}^T$ and $\mathbf{E}$ is $n \times k$. (Do not factor any other matrices, do not use the inverse of any matrices, and make the algorithm as efficient as you can.)

3. (10) Use the Gerschgorin theorem to show that eigenvalues of $\mathbf{M}^{-1}\mathbf{A}$ are close to those of $\mathbf{I}$ if $\|\mathbf{N}\| < \epsilon$ and ϵ is small enough. (You may use any matrix norm that you find convenient.) Under this assumption, what can you say about the convergence of cg using $\mathbf{M}$ as a preconditioner for $\mathbf{A}$? How small is "small enough"?

4. Let $\mathbf{T}$ be a *Toeplitz matrix*. Such a matrix is specified by a set of parameters $\boldsymbol{\beta}$ so that $t_{ij} = \beta_{i-j}$, $i, j = 1, \ldots, n$.

A *circulant matrix* is defined by specifying the first row of the matrix. Then any other row is equal to the preceding row, right circularly shifted by 1. For example,

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

is a circulant matrix. (Note that every circulant is a Toeplitz matrix, but the converse is not true.) One nice property of a circulant matrix $\mathbf{C}$ is that its eigenvectors are the columns of the Fourier transform matrix $\mathbf{F}$ defined by

$$f_{k\ell} = \frac{1}{\sqrt{n}} e^{2\pi i k \ell},$$

$k, \ell = 1, \ldots, n$, and its eigenvalues are computed by taking $\mathbf{F}$ times the first column of $\mathbf{C}$. Using the discrete Fourier transform algorithm, multiplication by $\mathbf{F}$ can be performed in $O(n \log_2 n)$ operations if n is a power of two. (If n has a lot of small prime factors, then the multiplication is also fast.)

4a. (10) Given a Toeplitz matrix $\mathbf{T}$, construct a circulant matrix $\mathbf{C}$ of larger size that has $\mathbf{T}$ as its upper left block. Describe an algorithm for forming $\mathbf{Tp}$ for an arbitrary vector $\mathbf{p}$ in the time it takes to form $\mathbf{C}$ times a vector, and show how to make this time at most $O(N \log_2 N)$, where N is the smallest power of 2 that is larger than $2n$.

4b. (10) Suppose $\mathbf{A}$ is a symmetric positive definite tridiagonal Toeplitz matrix. Find a circulant preconditioner so that cg converges in at most 3 iterations.