AMSC 607 / CMSC 764 Homework 1, Fall 2008
Partial Solution

1. (3) At what rate does the sequence

$$\boldsymbol{e}_k = 1 + (0.5)^{2^k}$$

converge to 1?

Answer: The limit of the sequence is 1, and, for $k = 0, 1, \dots,$

$$\frac{(\boldsymbol{e}_{k+1} - 1)}{(\boldsymbol{e}_k - 1)^r} = \frac{.5^{2^{k+1}}}{.5^{r2^k}}.$$

The ratio is 1 when $r = 2$, so the convergence rate is quadratic with rate constant 1.

2. (5) Let $\boldsymbol{x} = \boldsymbol{S}\boldsymbol{z} + \boldsymbol{d}$, where $\boldsymbol{S}$ is a given $n \times n$ matrix and $\boldsymbol{d}$ is a given $n \times 1$ vector.

Suppose f is a function of n variables, and define $\hat{f}(\boldsymbol{z}) = f(\boldsymbol{x}) = f(\boldsymbol{S}\boldsymbol{z} + \boldsymbol{d})$. Write expressions for the gradient and Hessian of $\hat{f}$ with respect to the variables $\boldsymbol{z}$, using the gradient and Hessian of the function f. (Hint: compute $\partial \hat{f}/\partial z_j$ by using the chain rule and the values $\partial f/\partial x_i$ and $\partial x_i/\partial z_j$.)

Answer: The gradient at $\boldsymbol{z}$ is

$$\boldsymbol{S}^T \boldsymbol{g}(\boldsymbol{S}\boldsymbol{z} + \boldsymbol{d}),$$

where $\boldsymbol{g}$ is the gradient of f, and the Hessian matrix at $\boldsymbol{z}$ is

$$\boldsymbol{S}^T \boldsymbol{H}(\boldsymbol{S}\boldsymbol{z} + \boldsymbol{d})\boldsymbol{S},$$

where $\boldsymbol{H}$ is the Hessian matrix of f. We'll use these expressions in our study of constrained optimization problems.

3. Consider the following problem: Find a value of γ so that the solution $\boldsymbol{p}$ to the linear system

$$(\boldsymbol{H} + \gamma \boldsymbol{I})\boldsymbol{p} = -\boldsymbol{g}$$

satisfies $\|\boldsymbol{p}\|_2 = \delta$, where $\delta > 0$ is a given value.

Suppose we have factored $\boldsymbol{H} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T$, where $\boldsymbol{\Lambda}$ is a diagonal matrix with diagonal elements λ_i and $\boldsymbol{U}$ is orthogonal, so that $\boldsymbol{U}\boldsymbol{U}^T = \boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}$.

3a. (5) Write the solution $\boldsymbol{p}$ to the linear system in terms of $\boldsymbol{g}$, γ, $\boldsymbol{U}$, $\boldsymbol{u}_i$, and λ_i, where $\boldsymbol{u}_i$ is the ith column of $\boldsymbol{U}$. (Hint: Remember that scalars like γ commute with matrices.)

Answer:

$$\begin{aligned} -\boldsymbol{g} = (\boldsymbol{H} + \gamma \boldsymbol{I})\boldsymbol{p} &= (\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T + \gamma \boldsymbol{I})\boldsymbol{p} \\ &= (\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T + \gamma \boldsymbol{U}\boldsymbol{U}^T)\boldsymbol{p} \\ &= \boldsymbol{U}(\boldsymbol{\Lambda} + \gamma \boldsymbol{I})\boldsymbol{U}^T\boldsymbol{p}. \end{aligned}$$

Let $\boldsymbol{s} = \boldsymbol{U}^T\boldsymbol{p}$ and $\boldsymbol{c} = -\boldsymbol{U}^T\boldsymbol{g}$. We multiply our equation by $\boldsymbol{U}^T$ to obtain

$$(\boldsymbol{\Lambda} + \gamma \boldsymbol{I})\boldsymbol{s} = \boldsymbol{c}.$$

Therefore, for $j = 1, \ldots, n$,

$$s_j = \frac{c_j}{\lambda_j + \gamma}.$$

Notice that $c_j = -\boldsymbol{u}_j^T\boldsymbol{g}$, and

$$\begin{aligned} \boldsymbol{p} &= \boldsymbol{U}\boldsymbol{s} \\ &= \sum_{j=1}^{n} s_j \boldsymbol{u}_j \\ &= \sum_{j=1}^{n} \frac{c_j}{\lambda_j + \gamma}\boldsymbol{u}_j. \end{aligned}$$

3ba. (2) Show that, for any vector $\boldsymbol{w}$, $\|\boldsymbol{w}\|_2 = \|\boldsymbol{U}^T\boldsymbol{w}\|_2$.

Answer: $\|\boldsymbol{U}^T\boldsymbol{w}\|_2^2 = (\boldsymbol{U}^T\boldsymbol{w})^T(\boldsymbol{U}^T\boldsymbol{w}) = \boldsymbol{w}^T\boldsymbol{U}\boldsymbol{U}^T\boldsymbol{w} = \boldsymbol{w}^T\boldsymbol{w} = \|\boldsymbol{w}\|_2^2$.

3bb. (5) Use 3ba to write an expression for $\|\boldsymbol{p}\|_2$ in terms of $\boldsymbol{g}$, γ, $\boldsymbol{U}$, and λ_i.

Answer: The previous result means that $\|\boldsymbol{p}\| = \|\boldsymbol{s}\|$, so

$$\begin{aligned} \|\boldsymbol{p}\|^2 &= \sum_{j=1}^{n} s_j^2 \\ &= \sum_{j=1}^{n} \frac{c_j^2}{(\lambda_j + \gamma)^2} \end{aligned}$$

$$= \sum_{j=1}^{n} \frac{(\boldsymbol{u}_j^T \boldsymbol{g})^2}{(\lambda_j + \gamma)^2}.$$

3c. (5) Find an interval for γ on which $\|\boldsymbol{p}\|_2$ is monotonically decreasing. (Hint: Remember that some of the λ_i might be negative.)

Answer: Notice that the expression in the previous answer goes to infinity when $\gamma = -\lambda_j$. But once γ is bigger than the absolute values of all of the negative eigenvalues of $\boldsymbol{A}$, then each of the terms in the summation is decreasing with γ, so the entire expression is monotonically decreasing with γ. Therefore, one such interval stretches from $\max(0, -\lambda_{min}(\boldsymbol{A}))$ to infinity, where $\lambda_{min}(\boldsymbol{A})$ is the smallest eigenvalue of $\boldsymbol{A}$.

3d. (5) Describe how you could use MATLAB's `fzero` to find a value of γ for which $\|\boldsymbol{p}\|_2 = \delta$ (if such a value exists). What initial interval would you give `zeroin`?

Answer: We use `fzero` to find a solution to $t(\gamma) \equiv \|\boldsymbol{p}\|^2 - \delta^2 = 0$, where

$$t(\gamma) = \sum_{j=1}^{n} \frac{(\boldsymbol{u}_j^T \boldsymbol{g})^2}{(\lambda_j + \gamma)^2} - \delta^2.$$

Evaluating this function is quite inexpensive, especially if we precompute (once only) the numerators.

Since

$$\sum_{j=1}^{n} \frac{(\boldsymbol{u}_j^T \boldsymbol{g})^2}{(\lambda_j + \gamma)^2} \leq n \frac{\max_j (\boldsymbol{u}_j^T \boldsymbol{g})^2}{\min_j (\lambda_j + \gamma)^2},$$

we can find an upper limit for our interval by solving

$$n \frac{\max_j (\boldsymbol{u}_j^T \boldsymbol{g})^2}{\min_j (\lambda_j + \gamma)^2} = \delta^2.$$

For the lower limit, it is probably easiest to use 0, if $\boldsymbol{A}$ is positive definite, or a number just bigger than $-\lambda_{min}(\boldsymbol{A})$, if $\boldsymbol{A}$ has a negative eigenvalue: for example, $-\lambda_{min}(\boldsymbol{A})(1 + \epsilon)$, where $1 + \epsilon$ is the floating-point number next to 1.

It is not a good idea to initialize `fzero` with points that give infinite function values.

In Homework 2, you will write a program, using this algorithm to solve minimization problems.