Examples of polynomial interpolation

For simplicity, we will take $n = 4$ in these examples.

Given: 4 data points

Find: a polynomial of degree 3 that satisfies the four conditions.

These notes illustrate the computational process of constructing an interpolating polynomial using the Newton basis.

Formulas

Given the four conditions $(z_0, f_0), \ldots, (z_3, f_3)$, the polynomial is

$$p(x) = f[z_0] + f[z_0,z_1](x-z_0) + f[z_0, z_1, z_2](x-z_0)(x-z_1)$$

$$+ f[z_0, z_1, z_2, z_3](x-z_0)(x-z_1)(x-z_2),$$

where $f[z_i] = f_i$ and

$$f[z_0,\ldots z_k] = (f[z_0,\ldots,z_{k-1}] - f[z_1,\ldots,z_k])/(z_0-z_k).$$

Example:

Suppose that we want

$p(1) = -5$	$z_0 = 1, f_0 = -5$
$p(2) = -3$	$z_1 = 2, f_1 = -3$
$p(3) = 2$	$z_2 = 3, f_2 = 2$
$p(4) = 4$	$z_3 = 4, f_3 = 4$

Constuct the first column

Divided difference table: $z_0=1, z_1=2, z_2=3, z_3=4$.

$f[z_0]=-5$

$f[z_1]=-3$

$f[z_2]= 2$

$f[z_3]= 4$

Construct the second column

Divided difference table: $z_0=1$, $z_1=2$, $z_2=3$, $z_3=4$.

$f[z_0]=-5$

$f[z_1]=-3$ $\quad \boxed{f[z_0,z_1]=\frac{-5-(-3)}{1-2}=2}$

$f[z_2]=2$

$f[z_3]=4$

Construct the second column

Divided difference table: $z_0=1$, $z_1=2$, $z_2=3$, $z_3=4$.

$f[z_0]=-5$

$f[z_1]=-3$ $\quad f[z_0,z_1]=\frac{-5-(-3)}{1-2}=2$

$f[z_2]=2$ $\quad f[z_1,z_2]=5$

$f[z_3]=4$ $\quad f[z_2,z_3]=2$

Construct the third column

Divided difference table: $z_0=1$, $z_1=2$, $z_2=3$, $z_3=4$.

$f[z_0]=-5$

$f[z_1]=-3$ $\quad f[z_0,z_1]=2$

$f[z_2]=2$ $\quad f[z_1,z_2]=5$ $\quad \boxed{f[z_0,z_1,z_2]=\frac{2-5}{1-3}=3/2}$

$f[z_3]=4$ $\quad f[z_2,z_3]=2$

Construct the third column

Divided difference table: $z_0=1$, $z_1=2$, $z_2=3$, $z_3=4$.

$f[z_0]=-5$

$f[z_1]=-3$ $\quad f[z_0,z_1]=2$

$f[z_2]=2$ $\quad f[z_1,z_2]=5$ $\quad f[z_0,z_1,z_2]=\frac{2-5}{1-3}=3/2$

$f[z_3]=4$ $\quad f[z_2,z_3]=2$ $\quad f[z_1,z_2,z_3]=-3/2$

Construct the last column

Divided difference table: $z_0=1$, $z_1=2$, $z_2=3$, $z_3=4$.

$f[z_0]=-5$

$f[z_1]=-3$ $\quad f[z_0,z_1] = 2$

$f[z_2]= 2$ $\quad f[z_1,z_2] = 5$ $\quad f[z_0,z_1,z_2] = 3/2$

$f[z_3]= 4$ $\quad f[z_2,z_3] = 2$ $\quad f[z_1,z_2,z_3] =-3/2$ $\quad f[z_0,z_1,z_2,z_3]= \dfrac{3/2+3/2}{1-4} =-1$

The resulting table

Divided difference table: $z_0=1$, $z_1=2$, $z_2=3$, $z_3=4$.

$f[z_0]=-5$

$f[z_1]=-3$ $\quad f[z_0,z_1] = 2$

$f[z_2]= 2$ $\quad f[z_1,z_2] = 5$ $\quad f[z_0,z_1,z_2] = 3/2$

$f[z_3]= 4$ $\quad f[z_2,z_3] = 2$ $\quad f[z_1,z_2,z_3] = -3/2$ $\quad f[z0,z_1,z_2,z_3]= -1$

The entries we need:

Divided difference table: $z_0=1$, $z_1=2$, $z_2=3$, $z_3=4$.

$\boxed{f[z_0]=-5}$

$f[z_1]=-3$ $\quad \boxed{f[z_0,z_1] = 2}$

$f[z_2]= 2$ $\quad f[z_1,z_2] = 5$ $\quad \boxed{f[z_0,z_1,z_2] = 3/2}$

$f[z_3]= 4$ $\quad f[z_2,z_3] = 2$ $\quad f[z_1,z_2,z_3] =-3/2$ $\quad \boxed{f[z_0,z_1,z_2,z_3]= -1}$

Resulting polynomial

Given the four conditions $(z_0, f_0), \dots, (z_3, f_3)$, the polynomial is

$$p(x) = f[z_0] + f[z_0,z_1](x-z_0) + f[z_0, z_1, z_2](x-z_0)(x-z_1)$$

$$+ f[z_0, z_1, z_2, z_3]\,(x-z_0)(x-z_1)(x-z_2)\,,$$

$$= -5 + 2\,(x-1) + 3/2\,(x-1)(x-2) - (x-1)(x-2)(x-3)$$