AMSC/CMSC 660 Quiz 3 , Fall 2003

1. (10) Suppose we have a matrix A of dimension $n \times n$ of rank $n-1$. Give two numerically stable ways to find a vector z so that $Az = 0$.

Answer: Some possibilities:

- Any right eigenvector u of A correponding to a zero eigenvalue satisfies $Au = 0u = 0$. With roundoff, the computed eigenvalue will not be exactly zero, so we can choose the eigenvector of A corresponding to the smallest magnitude eigenvalue.
- Similarly, if v is a right singular vector of A corresponding to a zero singular value, then $Av = 0$, so choose a singular vector corresponding to the smallest singular value.
- Let e_n be the vector with a 1 in position n and zeros elsewhere. If we perform a rank-revealing QR decomposition of A^T, so that $A^T P = QR$, and let q_n be the last column of Q, then $q_n^T A^T P = q_n^T QR = e_n^T R = r_{nn} e_n^T = 0$. Multiplying through by P^{-1} we see that $Aq_n = 0$, so choose $z = q_n$.

2. (10) Recall the Gram-Schmidt algorithm:

Set $r_{11} = \|a_1\|$.
Set $q_1 = a_1 / r_{11}$.
for $k = 1, \ldots n-1$,

Set $q_{k+1} = a_{k+1}$.
for $i = 1, \ldots, k$,

$r_{i,k+1} = q_{k+1}^T q_i$
$q_{k+1} = q_{k+1} - r_{i,k+1} q_i$

end for
$r_{k+1,k+1} = \|q_{k+1}\|$
$q_{k+1} = q_{k+1} / r_{k+1,k+1}$

end for

Show that $q_i^T q_k = 0$ for $i < k$.

Answer: A proof by finite induction. Note that after we finish the iteration $k = 1$, we have $q_{k+1} = q_{k+1} - r_{1,k+1} q_1$, so

$$q_1^T q_{k+1} = q_1^T q_{k+1} - r_{1,k+1} q_1^T q_1 = 0$$

by the definition of $r_{1,k+1}$ and the fact that $q_1^T q_1 = 1$.
Assume that after we finish iteration $i = j - 1$, for a given value of k, we have $q_\ell^T q_{k+1} = 0$ for $\ell \leq j - 1$ and $q_j^T q_\ell = 0$ for $j < \ell \leq k$. After we finish iteration $i = j$ for that value of k, we have $q_j^T q_{k+1} = 0$ by the same argument we used above, and we also have that $q_\ell^T q_{k+1} = 0$, for $\ell \leq j - 1$, since all we have done to q_{k+1} is to add a multiple of q_j to it, and q_j is orthogonal to q_ℓ. Thus, after iteration j, $q_j^T q_{k+1} = 0$ for $\ell \leq j$, and the induction is complete when $j = k$ and $k = n - 1$.