AMSC/CMSC 660 Quiz 2 , Fall 2004

1. Consider the following algorithm for generating random numbers:

> Take 5 papers and number them 0 to 4. Put them in a box, and draw one at random. Record the resulting number, and put the paper back in the box.

1a. (5) What is the probability density function for the distribution of samples drawn using this algorithm?

Answer: $p(x) = 1/5$ if $x = 0, 1, 2, 3, 4$. $p(x) = 0$ otherwise.

1b. (5) What are the mean and variance of the distribution?

Answer: Mean $= (0 + 1 + 2 + 3 + 4)/5 = 2$.

Variance $= [(0-2)^2 + (1-2)^2 + (2-2)^2 + (3-2)^2 + (4-2)^2]/5$
$= [4 + 1 + 0 + 1 + 4]/5 = 2$.

2. (10) Suppose we estimate the integral

$$I = \int_0^1 x^3 dx = \int_0^1 \left(\frac{x^2}{2}\right)(2x)dx$$

using Monte-Carlo integration (n samples) with importance sampling. (A silly example, but I hope it contributes to your understanding.)

2a. What will the variance in the estimates be if we let $p_1(x) = 1$ and choose the sample points for $f_1(x) = x^3$ according to the distribution $p_1(x)$?

Answer: The expected value is $I = 1/4$, and the variance is σ^2/n, where

$$\begin{aligned} \sigma^2 &= \int_0^1 (x^3 - 1/4)^2 dx \\ &= \int_0^1 x^6 - x^3/2 + 1/16 \; dx \\ &= 1/7 - 1/8 + 1/16 = .0804. \end{aligned}$$

2b. What will the variance in the estimates be if we let $p_2(x) = 2x$ and choose the sample points for $f_2(x) = x^2/2$ according to the distribution $p_2(x)$?

Answer: The expected value is $I = 1/4$, and the variance is σ^2/n, where

$$\begin{aligned} \sigma^2 &= \int_0^1 (x^2/2 - 1/4)^2 2x \, dx \\ &= \int_0^1 2x^5/4 - x^3/2 + x/8 \, dx \\ &= 1/12 - 1/8 + 1/16 = .0208. \end{aligned}$$

Notice that the variance has been reduced by a factor of approx. 4.