AMSC/CMSC 660 Quiz 5 , Fall 2004

1. (10) Suppose we have factored $A = LU$ and now we need to solve a linear system $(A - ZV^T)x = b$, where Z and V have dimension $n \times k$ and k is much less than n. Write Matlab code to do this correctly (5 points) and efficiently (5 points). You might want to use the Sherman-Morrison-Woodbury Formula:

$$(A - ZV^T)^{-1} = A^{-1} + A^{-1}Z(I - V^T A^{-1} Z)^{-1} V^T A^{-1}$$

Answer: We use several facts to get an algorithm that is $O(kn^2)$ instead of $O(n^3)$:

- $x = (A - ZV^T)^{-1}b = (A^{-1} + A^{-1}Z(I - V^T A^{-1}Z)^{-1}V^T A^{-1})b$.
- Forming A^{-1} from LU takes $O(n^3)$ operations, but forming $A^{-1}b$ as $U\backslash(L\backslash b)$ uses forward and backward substitution and just takes $O(n^2)$.
- $(I - V^T A^{-1} Z)$ is only $k \times k$, so factoring it is cheap: $O(k^3)$.
- Matrix multiplication is associative.

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y = U \ (L \ b);
Zh = U \ (L \ Z);
t = (eye(k) - V'*Zh) \ (V'*y);
x = y + Zh*t;
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2. (10) Let $f(x) = e^{x_1+x_2}x_1 + x_2^2$ and consider the point $x_1 = 1$, $x_2 = 0.3863$. Compute the Newton direction and determine whether it is downhill.
Helpful Fact: $e^{1.3863} = 4.0000$.

Answer:

$$g(x) = \begin{bmatrix} e^{x_1+x_2}(1+x_1) \\ e^{x_1+x_2}x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4.7726 \end{bmatrix};$$

$$H(x) = \begin{bmatrix} e^{x_1+x_2}(2+x_1) & e^{x_1+x_2}(1+x_1) \\ e^{x_1+x_2}(1+x_1) & e^{x_1+x_2}x_1 + 2 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 8 & 6 \end{bmatrix}$$

Now $\det(H) = 72 - 64 = 8$, so

$$p = -H^{-1}g = \frac{1}{8}\begin{bmatrix} 6 & -8 \\ -8 & 12 \end{bmatrix}\begin{bmatrix} -8 \\ -4.7726 \end{bmatrix} = \begin{bmatrix} -1.2274 \\ 0.8411 \end{bmatrix}$$

(We'd never use this inverse formula on a computer, except possibly for 2x2 matrices. Gauss elimination is generally better:

$$L = \begin{bmatrix} 1 & 0 \\ 2/3 & 1 \end{bmatrix}, U = \begin{bmatrix} 12 & 8 \\ 0 & 2/3 \end{bmatrix}.)$$

Note that $p^T g = -5.8050 < 0$, so the direction is downhill.