AMSC/CMSC 660 Quiz 2 , Fall 2006

1. (10) Suppose we have factored the $m \times n$ matrix $\mathbf{A} = \mathbf{QR}$ $(m \geq n)$, and let $\hat{\mathbf{x}}$ be the solution to the least squares problem

$$\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|.$$

Show that $\|\mathbf{A}\hat{\mathbf{x}} - \mathbf{b}\|^2 = \|\mathbf{c}_2\|^2$, where $\mathbf{c}_2$ is the vector consisting of the last $m - n$ components of $\mathbf{Q}^*\mathbf{b}$.

Answer: (Note that we must assume that $\mathbf{A}$ is full rank.)

Define

$$\mathbf{c} = \mathbf{Q}^*\mathbf{b} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix}, \mathbf{R} = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{bmatrix}$$

where $\mathbf{c}_1$ is $n \times 1$, $\mathbf{c}_2$ is $(m-n) \times 1$, $\mathbf{R}_1$ is $n \times n$, and $\mathbf{0}$ is $(m-n) \times n$. Then

$$\begin{aligned} \|\mathbf{Ax} - \mathbf{b}\|^2 &= \|\mathbf{Q}^*(\mathbf{Ax} - \mathbf{b})\|^2 \\ &= \|\mathbf{Rx} - \mathbf{c}\|^2 \\ &= \|\mathbf{R}_1\mathbf{x} - \mathbf{c}_1\|^2 + \|\mathbf{0x} - \mathbf{c}_2\|^2 \\ &= \|\mathbf{R}_1\mathbf{x} - \mathbf{c}_1\|^2 + \|\mathbf{c}_2\|^2. \end{aligned}$$

To minimize this quantity, we make the first term zero by taking $\mathbf{x}$ to be the solution to the $n \times n$ linear system $\mathbf{R}_1\mathbf{x} = \mathbf{c}_1$, so we see that the minimum value of $\|\mathbf{Ax} - \mathbf{b}\|$ is $\|\mathbf{c}_2\|$.

Note: The derivation above is based on three fundamental facts:

- Minimizing the norm of $\mathbf{Ax} - \mathbf{b}$ gives the same solution as minimizing the square of the norm.
- For any vector $\mathbf{y}$ and any unitary matrix $\mathbf{Q}$ the norm of the vector is invariant under multiplication by $\mathbf{Q}$, so $\|\mathbf{y}\| = \|\mathbf{Q}^*\mathbf{y}\|$.
- Suppose we partition the vector $\mathbf{y}$ into two pieces:

  $$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}.$$

  Then $\|\mathbf{y}\|^2 = \|\mathbf{y}_1\|^2 + \|\mathbf{y}_2\|^2$.

2. (10) Write a column-oriented algorithm to solve $\mathbf{Ux} = \mathbf{b}$ where $\mathbf{U}$ is an $n \times n$ nonsingular upper triangular matrix. (If you can't do this, you can get 5 points for any correct algorithm to solve this problem, but you may not use the backslash operator or an inverse matrix.)

Answer: (Note that this is like the 4th exercise.)

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x = b;
for j=n:-1:1,
   x(j) = x(j) / U(j,j);
   x(1:j-1) = x(1:j-1) - U(1:j-1,j)*x(j);
end
```