AMSC/CMSC 660 Quiz 3 , Fall 2006

1. (10) Suppose we have factored $\mathbf{A} = \mathbf{LU}$ and now we need to solve a linear system $(\mathbf{A} - \mathbf{Z}\mathbf{V}^T)\mathbf{x} = \mathbf{b}$, where $\mathbf{Z}$ and $\mathbf{V}$ have dimension $n \times k$ and k is much less than n. Write MATLAB code to do this accurately and efficiently. You might want to use the Sherman-Morrison-Woodbury formula

$$(\mathbf{A} - \mathbf{Z}\mathbf{V}^T)^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{Z}(\mathbf{I} - \mathbf{V}^T\mathbf{A}^{-1}\mathbf{Z})^{-1}\mathbf{V}^T\mathbf{A}^{-1}.$$

(Don't use matrix inverses!)

Answer: This is Exercise 14.

We use several facts to get an algorithm that is $O(kn^2)$ instead of $O(n^3)$:

- $\mathbf{x} = (\mathbf{A} - \mathbf{Z}\mathbf{V}^T)^{-1}\mathbf{b} = (\mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{Z}(\mathbf{I} - \mathbf{V}^T\mathbf{A}^{-1}\mathbf{Z})^{-1}\mathbf{V}^T\mathbf{A}^{-1})\mathbf{b}$.
- Forming $\mathbf{A}^{-1}$ from $\mathbf{LU}$ takes $O(n^3)$ operations, but forming $\mathbf{A}^{-1}\mathbf{b}$ as $\mathbf{U}\backslash(\mathbf{L}\backslash\mathbf{b})$ uses forward and backward substitution and just takes $O(n^2)$.
- $(\mathbf{I} - \mathbf{V}^T\mathbf{A}^{-1}\mathbf{Z})$ is only $k \times k$, so factoring it is cheap: $O(k^3)$.
- Matrix multiplication is associative.

The MATLAB code is:

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y = U \ (L \ b);
Zh = U \ (L \ Z);
t = (eye(k) - V'*Zh) \ (V'*y);
x = y + Zh*t;
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2. (10) Denote the SVD of the 2×2 matrix $\mathbf{A}$ by $\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$.

(a) Express the solution to the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ as $\mathbf{x} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2$ where $\mathbf{V} = [\mathbf{v}_1,\ \mathbf{v}_2]$.

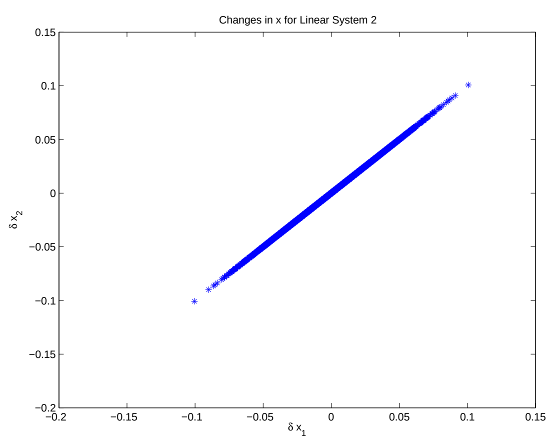
(b) Consider the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} 1+\delta & \delta-1 \\ \delta-1 & 1+\delta \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

and $\delta = 0.002$, and suppose we compute the solution to the nearby systems

$$(\mathbf{A} + \mathbf{E}^{(i)})\mathbf{x}^{(i)} = \mathbf{b}$$

for $i = 1, \ldots, 1000$, where the elements of $\mathbf{E}^{(i)}$ are independent and normally distributed with mean 0 and standard deviation $\tau = .0001$. Using part (a), explain why the resulting solutions all fall near a straight line, as shown in the figure.



Answer:

(a) Since $\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\mathbf{x} = \mathbf{b}$, we have

$$\mathbf{x} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T\mathbf{b}.$$

If we let $\mathbf{c} = \mathbf{U}^T\mathbf{b}$, then $\alpha_1 = c_1/\sigma_1$, and $\alpha_2 = c_2/\sigma_2$.

(b) Here is one way to look at it. You saw in the homework that this system is very ill-conditioned – the two equations lie almost on top of each other. The condition number is the ratio of the largest singular value to the smallest, so this must be large. In other words, σ_2 is quite small compared to σ_1.

For the perturbed problems,

$$\mathbf{A}\mathbf{x}^{(i)} = \mathbf{b} - \mathbf{E}^{(i)}\mathbf{x}^{(i)},$$

so it is as if we solve the linear system with a slightly perturbed right-hand side. So, letting $\mathbf{f}^{(i)} = \mathbf{U}^T\mathbf{E}^{(i)}\mathbf{x}^{(i)}$, the computed solution is

$$\mathbf{x}^{(i)} = \alpha_1^{(i)}\mathbf{v}_1 + \alpha_2^{(i)}\mathbf{v}_2$$

with

$$\alpha_1^{(i)} = (c_1 + f_1^{(i)})/\sigma_1, \quad \alpha_2^{(i)} = (c_2 + f_2^{(i)})/\sigma_2.$$

From the figure, we know that $\mathbf{f}^{(i)}$ must be small, so $\alpha_1^{(i)} \approx \alpha_1$. But because σ_2 is close to zero, $\alpha_2^{(i)}$ can be quite different from α_2, so the solutions lie almost on a straight line in the direction $\mathbf{v}_2$.