1. (10) Write MATLAB code using rand to generate a random number from the following distribution:

The probability that the number is 0 is 0.6.
The probability that the number is 1 is 0.4.

(In other words, if \( p(x) \) is the probability density function, then \( p(0) = 0.6 \) and \( p(1) = 0.4 \).)

**Answer:**

% In this code,
% z is a sample from a uniform distribution on [0,1].
% y is a sample from the desired distribution.

```matlab
z = rand(1);
if (z < .6)
    y = 0;
else
    y = 1;
end
```
2. (10) Write MATLAB code to compute the volume of the unit sphere \( x_1^2 + x_2^2 + x_3^2 \leq 1 \) using Monte Carlo integration. (You may use any of our three methods, although I suggest not using importance sampling because it is harder to write down.)

**Answer:** Here are implementations of “Method 1” (5 lines of MATLAB code) and “Method 2” (8 lines of code).

```matlab
% "Method 1": We compute the volume of the unit sphere as 8 times the volume of the piece of it in the 1st orthant (x_1 >= 0, x_2 >= 0, x_3 >= 0). Call this piece Omega.

% We can estimate the volume of Omega by uniformly sampling points in the unit box and counting the proportion of them that are also in Omega.

% Let N be the number of sample points.

N = 1000 % as an example

% Construct N samples, 3 coordinates each.

x = rand(N,3);

% Compute x_1^2 + x_2^2 + x_3^2 - 1 for each of them. (This uses the vector capabilities of Matlab and is faster than doing it component by component.)

y = x(:,1).^2 + x(:,2).^2 + x(:,3).^2 - 1;

% Count the number of points in the region, and divide by N to get the estimate of the volume of Omega.

Omega_vol = sum(y < 0) / N;

% Multiply by 8 to estimate the volume of the sphere.

Sphere_vol = 8 * Omega_vol;
```
Or:

"Method 2": We can compute the volume of the unit sphere
as 2 times the integral from \( y = -1 \) to \( y = 1 \) of
the integral from \( x = -\sqrt{1-y^2} \) to \( x = \sqrt{1-y^2} \)
of \( \sqrt{1-x^2-y^2} \). (See pointer on p. 182)

In other words, we need the integral of \( \sqrt{1-x^2-y^2} \)
over the domain that is the unit circle.

Let \( N \) be the desired number of sample points.

\[ N = 1000 \] as an example

Construct (approx.) \( N \) samples, uniformly distributed in the unit
quarter circle.
(Since the volume of the unit box is 1 and the volume of the quarter
circle is \( \pi/4 \), we need to sample about \( 4N/\pi \) points in the box
to get about \( N \) in the quarter circle.)

\[
x = \text{rand}(\text{ceil}(4*\text{N}/\pi),2); \quad \%	ext{These are uniformly distributed in the square.}
\]
\[
\text{test} = x(:,1).^2 + x(:,2).^2 - 1;
\]
\[
[\text{good_indices},z] = \text{find}(\text{test} < 0);
\]
\[
x = x(\text{good_indices},:); \quad \%	ext{These are uniformly distributed in the quarter circle.}
\]

\%
Find the average function value.

\[
z = \sqrt{1-x(:,1).^2 - x(:,2).^2};
\]
\[
\text{averagez} = \text{sum}(z)/\text{length}(z);
\]

\%
The estimate of the integral is 8 times \( \text{averagez} \) times the area of the
quarter circle.

\[
\text{Sphere_vol} = 2 * \text{averagez} * \pi;
\]