AMSC/CMSC 660 Quiz 5 , Fall 2006

1a. (5) Verify that the BFGS matrix

$$\mathbf{B}^{(k+1)} = \mathbf{B}^{(k)} - \frac{\mathbf{B}^{(k)}\mathbf{s}^{(k)}\mathbf{s}^{(k)T}\mathbf{B}^{(k)}}{\mathbf{s}^{(k)T}\mathbf{B}^{(k)}\mathbf{s}^{(k)}} + \frac{\mathbf{y}^{(k)}\mathbf{y}^{(k)T}}{\mathbf{y}^{(k)T}\mathbf{s}^{(k)}}.$$

satisfies the secant condition: $\mathbf{B}^{(k+1)}\mathbf{s}^{(k)} = \mathbf{y}^{(k)}$.

1b. (5) Define $\mathbf{s}^{(k)}$ and $\mathbf{y}^{(k)}$. Why are quasi-Newton matrices designed to satisfy the secant condition?

Answer:

1a.

$$\begin{aligned}
\mathbf{B}^{(k+1)}\mathbf{s}^{(k)} &= \mathbf{B}^{(k)}\mathbf{s}^{(k)} - \frac{\mathbf{B}^{(k)}\mathbf{s}^{(k)}\mathbf{s}^{(k)T}\mathbf{B}^{(k)}}{\mathbf{s}^{(k)T}\mathbf{B}^{(k)}\mathbf{s}^{(k)}}\mathbf{s}^{(k)} + \frac{\mathbf{y}^{(k)}\mathbf{y}^{(k)T}}{\mathbf{y}^{(k)T}\mathbf{s}^{(k)}}\mathbf{s}^{(k)} \\
&= \mathbf{B}^{(k)}\mathbf{s}^{(k)} - \frac{\mathbf{B}^{(k)}\mathbf{s}^{(k)}\mathbf{s}^{(k)T}\mathbf{B}^{(k)}\mathbf{s}^{(k)}}{\mathbf{s}^{(k)T}\mathbf{B}^{(k)}\mathbf{s}^{(k)}} + \frac{\mathbf{y}^{(k)}\mathbf{y}^{(k)T}\mathbf{s}^{(k)}}{\mathbf{y}^{(k)T}\mathbf{s}^{(k)}} \\
&= \mathbf{B}^{(k)}\mathbf{s}^{(k)} - \mathbf{B}^{(k)}\mathbf{s}^{(k)}\frac{\mathbf{s}^{(k)T}\mathbf{B}^{(k)}\mathbf{s}^{(k)}}{\mathbf{s}^{(k)T}\mathbf{B}^{(k)}\mathbf{s}^{(k)}} + \mathbf{y}^{(k)}\frac{\mathbf{y}^{(k)T}\mathbf{s}^{(k)}}{\mathbf{y}^{(k)T}\mathbf{s}^{(k)}} \\
&= \mathbf{B}^{(k)}\mathbf{s}^{(k)} - \mathbf{B}^{(k)}\mathbf{s}^{(k)} + \mathbf{y}^{(k)} \\
&= \mathbf{y}^{(k)}.
\end{aligned}$$

1b. For quadratic functions, we have $\mathbf{H}\mathbf{s}^{(k)} = \mathbf{y}^{(k)}$, where $\mathbf{s}^{(k)} = \mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}$ (the change in x) and $\mathbf{y}^{(k)} = \mathbf{g}^{(k+1)} - \mathbf{g}^{(k)}$ (the change in gradient). Therefore, we demand the same property from the Quasi-Newton matrix, since it forms a quadratic model for our function.

2. (10) Write a MATLAB program to apply 5 iterations of Newton's method to the problem

$$\min_{\mathbf{x}} (x_1 - 2)^4 + (x_2 + 1)^4 - x_1^2 x_2$$

with a steplength of 1 (i.e, step in the Newton direction without a linesearch) and with an initial starting guess of $\mathbf{x} = [1, 2]^T$.

Answer:

```
x = [1,2]';
for k=1:5,
  g = [4*(x(1)-2)^3 - 2*x(1)*x(2);
       4*(x(2)+1)^3 - x(1)^2];
  H = [12*(x(1)-2)^2-2*x(2), -2*x(1)
       -2*x(1), 12*(x(2)+1)^2];
  p = -H\g;
  x = x + p;
end
```