1. (10) Let \( f(x) = \frac{1}{2}x^T H x - x^T b \), where \( H \) and \( b \) are constant, independent of \( x \), and \( H \) is symmetric positive definite. Given vectors \( x^{(0)} \) and \( p^{(0)} \), find the value of the scalar \( \alpha \) that minimizes \( f(x^{(0)} + \alpha p^{(0)}) \).

**Answer:** Dropping superscripts for brevity, and taking advantage of symmetry of \( H \), we obtain

\[
f(x^{(0)} + \alpha p^{(0)}) = \frac{1}{2}(x + \alpha p)^T H (x + \alpha p) - (x + \alpha p)^T b
= \frac{1}{2} x^T H x - x^T b + \alpha p^T H x + \frac{1}{2} \alpha^2 p^T H p - \alpha p^T b.
\]

Differentiating with respect to \( \alpha \) we obtain

\[
p^T H x + \alpha p^T H p - p^T b = 0,
\]

so

\[
\alpha = \frac{p^T b - p^T H x}{p^T H p} = \frac{p^T r}{p^T H p},
\]

where \( r = b - H x \).

If we differentiate a second time, we find that the second derivative of \( f \) with respect to \( \alpha \) is \( p^T H p > 0 \) (when \( p \neq 0 \)), so we have found a minimizer.

**Note:** This is the formula for the step in the linear conjugate gradient algorithm.
2. (10) Consider the problem

\[ \min_{x} 5x_1^4 + x_1x_2 + 6x_2^2 \]

subject to the constraints \( x \geq 0 \) and \( x_1 - 2x_2 = 4 \). Formulate this problem as an unconstrained optimization problem using feasible directions and a barrier function.

**Answer:** The vector \([6,1]^T\) is a particular solution to \( x_1 - 2x_2 = 4 \), and the vector \([2,1]^T\) is a basis for the nullspace of the matrix \( A = [1, -2] \). (These choices are not unique, so there are many correct answers) Using our choices, any solution to the equality constraint can be expressed as

\[ x = \begin{bmatrix} 6 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} v = \begin{bmatrix} 6 + 2v \\ 1 + v \end{bmatrix}. \]

Therefore, our problem is equivalent to

\[ \min_{v} 5(6 + 2v)^4 + (6 + 2v)(1 + v) + 6(1 + v)^2 \]

subject to

\[ 6 + 2v \geq 0, \]
\[ 1 + v \geq 0. \]

Using a log barrier function for these constraints, we obtain the unconstrained problem

\[ \min_{v} B_{\mu}(v) \]

where

\[ B_{\mu}(v) = 5(6 + 2v)^4 + (6 + 2v)(1 + v) + 6(1 + v)^2 - \mu \log(6 + 2v) - \mu \log(1 + v). \]

Notice that if \( 1 + v \geq 0 \), then \( 6 + 2v \geq 0 \). Therefore, the first log term can be dropped from \( B_{\mu}(v) \).