AMSC/CMSC 660 Quiz 6 , Fall 2006

1. (10) Let $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x} - \mathbf{x}^T\mathbf{b}$, where $\mathbf{H}$ and $\mathbf{b}$ are constant, independent of $\mathbf{x}$, and $\mathbf{H}$ is symmetric positive definite. Given vectors $\mathbf{x}^{(0)}$ and $\mathbf{p}^{(0)}$, find the value of the scalar α that minimizes $f(\mathbf{x}^{(0)} + \alpha\mathbf{p}^{(0)})$.

Answer: Dropping superscripts for brevity, and taking advantage of symmetry of $\mathbf{H}$, we obtain

$$\begin{aligned} f(\mathbf{x}^{(0)} + \alpha\mathbf{p}^{(0)}) &= \frac{1}{2}(\mathbf{x} + \alpha\mathbf{p})^T\mathbf{H}(\mathbf{x} + \alpha\mathbf{p}) - (\mathbf{x} + \alpha\mathbf{p})^T\mathbf{b} \\ &= \frac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x} - \mathbf{x}^T\mathbf{b} + \alpha\mathbf{p}^T\mathbf{H}\mathbf{x} + \frac{1}{2}\alpha^2\mathbf{p}^T\mathbf{H}\mathbf{p} - \alpha\mathbf{p}^T\mathbf{b}. \end{aligned}$$

Differentiating with respect to α we obtain

$$\mathbf{p}^T\mathbf{H}\mathbf{x} + \alpha\mathbf{p}^T\mathbf{H}\mathbf{p} - \mathbf{p}^T\mathbf{b} = 0,$$

so

$$\alpha = \frac{\mathbf{p}^T\mathbf{b} - \mathbf{p}^T\mathbf{H}\mathbf{x}}{\mathbf{p}^T\mathbf{H}\mathbf{p}} = \frac{\mathbf{p}^T\mathbf{r}}{\mathbf{p}^T\mathbf{H}\mathbf{p}},$$

where $\mathbf{r} = \mathbf{b} - \mathbf{H}\mathbf{x}$.

If we differentiate a second time, we find that the second derivative of f with respect to α is $\mathbf{p}^T\mathbf{H}\mathbf{p} > 0$ (when $\mathbf{p} \neq \mathbf{0}$), so we have found a minimizer.

Note: This is the formula for the step in the linear conjugate gradient algorithm.

2. (10) Consider the problem

$$\min_{\mathbf{x}} 5x_1^4 + x_1 x_2 + 6x_2^2$$

subject to the constraints $\mathbf{x} \geq \mathbf{0}$ and $x_1 - 2x_2 = 4$. Formulate this problem as an unconstrained optimization problem using feasible directions and a barrier function.

Answer: The vector $[6, 1]^T$ is a particular solution to $x_1 - 2x_2 = 4$, and the vector $[2, 1]^T$ is a basis for the nullspace of the matrix $\mathbf{A} = [1, -2]$. (These choices are not unique, so there are many correct answers) Using our choices, any solution to the equality constraint can be expressed as

$$\mathbf{x} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} v = \begin{bmatrix} 6 + 2v \\ 1 + v \end{bmatrix}.$$

Therefore, our problem is equivalent to

$$\min_{v} 5(6 + 2v)^4 + (6 + 2v)(1 + v) + 6(1 + v)^2$$

subject to

$$\begin{aligned} 6 + 2v &\geq 0, \\ 1 + v &\geq 0. \end{aligned}$$

Using a log barrier function for these constraints, we obtain the unconstrained problem

$$\min_{v} B_\mu(v)$$

where

$$B_\mu(v) = 5(6+2v)^4 + (6+2v)(1+v) + 6(1+v)^2 - \mu \log(6+2v) - \mu \log(1+v).$$

Notice that if $1 + v \geq 0$, then $6 + 2v \geq 0$. Therefore, the first log term can be dropped from $B_\mu(v)$.