1. (10) Write MATLAB code to apply 5 steps of Newton’s method to the problem

\[
\begin{align*}
  x^2y^3 + xy &= 2, \\
  2xy^2 + x^2y + xy &= 0,
\end{align*}
\]

starting at the point \(x = 5, \ y = 4\).

Answer:

\[
F(x) = \begin{bmatrix}
  x^2y^3 + xy - 2 \\
  2xy^2 + x^2y + xy
\end{bmatrix}
\]

and

\[
J(x) = \begin{bmatrix}
  2xy^3 + y & 3x^2y^2 + x \\
  2y^2 + 2xy + y & 4xy + x^2 + x
\end{bmatrix}.
\]

\[
x = [5;4];
\]

for \(i=1:5\),

\[
F = [x(1)^2*x(2)^3 + x(1)*x(2) - 2 \\
2*x(1)*x(2)^2 + x(1)^2*2*x(2) + x(1)*x(2)];
\]

\[
J = [2*x(1)*x(2)^3 + x(2), 3*x(1)^2*2*x(2)^2 + x(1) \\
2*x(2)^2 + 2*x(1)*x(2) + x(2), 4*x(1)*x(2) + x(1)^2 + x(1)];
\]

\[
x = x - J\backslash F;
\]
end
2. Consider using a homotopy method to solve the problem

\[ \mathbf{F}(\mathbf{x}) = \begin{bmatrix} x^2y^3 + xy - 2 \\ 2xy^2 + x^2y + xy \end{bmatrix} = \mathbf{0}. \]

Our homotopy function is

\[ \rho_\lambda(\lambda, \mathbf{x}) = \lambda \mathbf{F}(\mathbf{x}) + (1 - \lambda)(\mathbf{x} - \mathbf{a}), \]

where \( \mathbf{x} = [x, y]^T \).

(a) (4) Compute the Jacobian matrix for \( \rho_\lambda(\lambda, \mathbf{x}) \).

(b) (6) What needs to hold in order that the function \( \rho_\lambda \) is transversal to zero on its domain? Why is this likely to be true?

**Answer:**

(a) We compute the partial of \( \rho_\lambda(\lambda, \mathbf{x}) \) with respect to \( \lambda \):

\[ \mathbf{s} = \begin{bmatrix} x^2y^3 + xy - 2 - (x - a_1) \\ 2xy^2 + x^2y + xy - (y - a_2) \end{bmatrix}. \]

Then the Jacobian of \( \rho_\lambda \) is the \( 2 \times 3 \) matrix

\[ \mathbf{J}(\lambda, \mathbf{x}) = [ \mathbf{s}, (1 - \lambda)\mathbf{I} + \lambda \mathbf{J}(\mathbf{x}) ] \]

where \( \mathbf{J}(\mathbf{x}) \) is the matrix from Problem 1.

(b) In order for the function to be transversal to zero, the matrix \( \mathbf{J}(\lambda, \mathbf{x}) \) must be full rank (i.e., rank-2) at every point \( \lambda \in [0, 1), x, y \in (-\infty, \infty) \).

The matrix \( \mathbf{J}(\mathbf{x}) \) has two eigenvalues – call them \( \alpha_1 \) and \( \alpha_2 \). The matrix

\[ \mathbf{K} = (1 - \lambda)\mathbf{I} + \lambda \mathbf{J}(\mathbf{x}) \]

has eigenvalues \( (1 - \lambda) + \lambda \alpha_i \), so it is singular only if \( \lambda = 1/(1 - \alpha_1) \) or \( \lambda = 1/(1 - \alpha_2) \). Even if that happens, it is likely that the vector \( \mathbf{s} \) will point in a different direction, making the rank of \( \mathbf{J}(\lambda, \mathbf{x}) \) equal to 2.