AMSC/CMSC 660 Quiz 7 , Fall 2006

1. (10) Write Matlab code to apply 5 steps of Newton's method to the problem

$$\begin{aligned} x^2y^3 + xy &= 2\,, \\ 2xy^2 + x^2y + xy &= 0\,, \end{aligned}$$

starting at the point $x = 5$, $y = 4$.

Answer:

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} x^2y^3 + xy - 2 \\ 2xy^2 + x^2y + xy \end{bmatrix}$$

and

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} 2xy^3 + y & 3x^2y^2 + x \\ 2y^2 + 2xy + y & 4xy + x^2 + x \end{bmatrix}.$$

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x = [5;4];
for i=1:5,
    F = [x(1)^2*x(2)^3 + x(1)*x(2) -  2
         2*x(1)*x(2)^2 + x(1)^2*x(2) + x(1)*x(2)];
    J = [2*x(1)*x(2)^3 + x(2),  3*x(1)^2*x(2)^2 + x(1)
         2*x(2)^2 + 2*x(1)*x(2) + x(2),  4*x(1)*x(2) + x(1)^2 + x(1)];
    x = x - J\F;
end
```

2. Consider using a homotopy method to solve the problem

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} x^2y^3 + xy - 2 \\ 2xy^2 + x^2y + xy \end{bmatrix} = \mathbf{0}.$$

Our homotopy function is

$$\rho_{\mathbf{a}}(\lambda, \mathbf{x}) = \lambda \mathbf{F}(\mathbf{x}) + (1 - \lambda)(\mathbf{x} - \mathbf{a}),$$

where $\mathbf{x} = [x, y]^T$.

(a) (4) Compute the Jacobian matrix for $\rho_{\mathbf{a}}(\lambda, \mathbf{x})$.

(b) (6) What needs to hold in order that the function $\rho_{\mathbf{a}}$ is transversal to zero on its domain? Why is this likely to be true?

Answer:

(a) We compute the partial of $\rho_{\mathbf{a}}(\lambda, \mathbf{x})$ with respect to λ:

$$\mathbf{s} = \begin{bmatrix} x^2y^3 + xy - 2 - (x - a_1) \\ 2xy^2 + x^2y + xy - (y - a_2) \end{bmatrix}.$$

Then the Jacobian of ρ_a is the 2×3 matrix

$$\mathbf{J}(\lambda, \mathbf{x}) = \begin{bmatrix} \mathbf{s}, & (1 - \lambda)\mathbf{I} + \lambda \mathbf{J}(\mathbf{x}) \end{bmatrix}$$

where $\mathbf{J}(\mathbf{x})$ is the matrix from Problem 1.

(b) In order for the function to be transversal to zero, the matrix $\mathbf{J}(\lambda, \mathbf{x})$ must be full rank (i.e., rank-2) at every point $\lambda \in [0, 1)$, $x, y \in (-\infty, \infty)$.

The matrix $\mathbf{J}(\mathbf{x})$ has two eigenvalues – call them α_1 and α_2. The matrix $\mathbf{K} = (1 - \lambda)\mathbf{I} + \lambda \mathbf{J}(\mathbf{x})$ has eigenvalues $(1 - \lambda) + \lambda \alpha_i$, so it is singular only if $\lambda = 1/(1 - \alpha_1)$ or $\lambda = 1/(1 - \alpha_2)$. Even if that happens, it is likely that the vector $\mathbf{s}$ will point in a different direction, making the rank of $\mathbf{J}(\lambda, \mathbf{x})$ equal to 2.