1. (10) Recall that a Hamiltonian system is a system of ODEs for which there exists a scalar Hamiltonian function $H(y)$ so that $y' = D \nabla_y H(y)$ where $D$ is a block-diagonal matrix with blocks equal to

$$\begin{bmatrix}
0 & 1 \\
-1 & 0 
\end{bmatrix}.$$ 

Derive the Hamiltonian system for

$$H(y) = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{2}y_3^2 + \frac{1}{2}y_4^2 + \frac{1}{2}y_1^2 y_2^2 y_3^2 y_4^2.$$ 

Answer:

$$y' = D \nabla_y H(y)$$

$$= $$

$$\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0
\end{bmatrix}$$

$$\begin{bmatrix}
y_1 + y_1 y_2^2 y_3^2 y_4^2 \\
y_2 + y_1^2 y_2 y_3^2 y_4^2 \\
y_3 + y_1^2 y_2^2 y_3 y_4^2 \\
y_4 + y_1^2 y_2^2 y_3^2 y_4
\end{bmatrix}$$

$$= $$

$$\begin{bmatrix}
y_2 + y_1^2 y_2 y_3^2 y_4^2 \\
-(y_1 + y_1 y_2^2 y_3^2 y_4^2) \\
y_4 + y_1^2 y_2^2 y_3^2 y_4 \\
-(y_3 + y_1^2 y_2 y_3 y_4^2)
\end{bmatrix}$$
2. (10) Suppose we have used the Adams-Bashforth and Adams-Moulton methods of order 3 to form two estimates of \( y(t_{n+1}) \), the solution to a differential equation. These formulas are:

\[
y_{n+1}^{ab} = y_n + \frac{h}{12} (23f_n - 16f_{n-1} + 5f_{n-2}) \quad \text{error} : \frac{3h^4}{8} y^{(4)}(\eta).
\]

\[
y_{n+1}^{am} = y_n + \frac{h}{12} (5f_{n+1} + 8f_n - f_{n-1}) \quad \text{error} : -\frac{h^4}{24} y^{(4)}(\eta).
\]

How would you estimate the local error in the Adams-Moulton formula? How would you use that estimate to change \( h \) in order to keep the estimated local error less than a user-supplied local error tolerance \( \tau \) without taking steps smaller than necessary?

**Answer:** We know that if our old values are correct,

\[
y_{n+1}^{ab} - y(t_{n+1}) = \frac{3h^4}{8} y^{(4)}(\eta).
\]

\[
y_{n+1}^{am} - y(t_{n+1}) = -\frac{h^4}{24} y^{(4)}(\nu).
\]

Subtracting, we obtain

\[
y_{n+1}^{ab} - y_{n+1}^{am} = \frac{3h^4}{8} y^{(4)}(\eta) - (-\frac{h^4}{24} y^{(4)}(\nu))
\]

where \( \eta, \nu \) are in the interval containing \( y_{n+1}^{ab}, y_{n+1}^{am} \), and the true value. Since \( 3/8 + 1/24 = 10/24 \), the error in AM can be estimated as \( \epsilon = |y_{n+1}^{ab} - y_{n+1}^{am}|/10 \).

Now, if \( \epsilon > \tau \), we might reduce \( h \) by a factor of 2 and retake the step. If \( \epsilon << \tau \), we might double \( h \) in preparation for the next step (expecting that the local error might increase by a factor of \( 2^4 \)).