AMSC/CMSC 660 Quiz 9 , Fall 2006

1. (10) Recall that a Hamiltonian system is a system of ODEs for which there exists a scalar Hamiltonian function $H(\mathbf{y})$ so that $\mathbf{y}' = \mathbf{D}\nabla_{\mathbf{y}} H(\mathbf{y})$ where $\mathbf{D}$ is a block-diagonal matrix with blocks equal to

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Derive the Hamiltonian system for

$$H(\mathbf{y}) = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{2}y_3^2 + \frac{1}{2}y_4^2 + \frac{1}{2}y_1^2 y_2^2 y_3^2 y_4^2.$$

Answer:

$$\begin{aligned}
\mathbf{y}' &= \mathbf{D}\nabla_{\mathbf{y}} H(\mathbf{y}) \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 + y_1\, y_2^2\, y_3^2\, y_4^2 \\ y_2 + y_1^2\, y_2\, y_3^2\, y_4^2 \\ y_3 + y_1^2\, y_2^2\, y_3\, y_4^2 \\ y_4 + y_1^2\, y_2^2\, y_3^2\, y_4 \end{bmatrix} \\
&= \begin{bmatrix} y_2 + y_1^2\, y_2\, y_3^2\, y_4^2 \\ -(y_1 + y_1\, y_2^2\, y_3^2\, y_4^2) \\ y_4 + y_1^2\, y_2^2\, y_3^2\, y_4 \\ -(y_3 + y_1^2\, y_2^2\, y_3\, y_4^2) \end{bmatrix}
\end{aligned}$$

2. (10) Suppose we have used the Adams-Bashforth and Adams-Moulton methods of order 3 to form two estimates of $y(t_{n+1})$, the solution to a differential equation. These formulas are:

$$\begin{aligned} y_{n+1}^{ab} &= y_n + \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2}) \; error: \frac{3h^4}{8}y^{(4)}(\eta). \\ y_{n+1}^{am} &= y_n + \frac{h}{12}(5f_{n+1} + 8f_n - f_{n-1}) \; error: -\frac{h^4}{24}y^{(4)}(\eta). \end{aligned}$$

How would you estimate the local error in the Adams-Moulton formula? How would you use that estimate to change h in order to keep the estimated local error less than a user-supplied local error tolerance τ without taking steps smaller than necessary?

Answer: We know that if our old values are correct,

$$\begin{aligned} y_{n+1}^{ab} - y(t_{n+1}) &= \frac{3h^4}{8}y^{(4)}(\eta). \\ y_{n+1}^{am} - y(t_{n+1}) &= -\frac{h^4}{24}y^{(4)}(\nu). \end{aligned}$$

Subtracting, we obtain

$$y_{n+1}^{ab} - y_{n+1}^{am} = \frac{3h^4}{8}y^{(4)}(\eta) - (-\frac{h^4}{24}y^{(4)}(\nu))$$

where η, ν are in the interval containing y_{n+1}^{ab}, y_{n+1}^{am}, and the true value. Since $3/8+1/24 = 10/24$, the error in AM can be estimated as $\epsilon = |y_{n+1}^{ab} - y_{n+1}^{am}|/10$. Now, if $\epsilon > \tau$, we might reduce h by a factor of 2 and retake the step. If $\epsilon << \tau$, we might double h in preparation for the next step (expecting that the local error might increase by a factor of 2^4).