AMSC/CMSC 661 Quiz 7 , Spring 2005

1. (10) We want to solve the ordinary differential equation

$$u' + au = 0$$

where $u : \mathcal{R} \to \mathcal{R}$, $a > 0$ is a scalar, and $u(0)$ is given. Consider the numerical method

$$\frac{u^{n+1} - u^n}{k} + a\frac{u^{n+1} + u^n}{2} = 0$$

where $u^n \approx u(nk)$ and $k > 0$ is the time step. For what range of k values is the numerical method stable?

Answer: Rearranging, we get

$$(1 + ka/2)u^{n+1} = (1 - ka/2)u^n$$

so

$$u^{n+1} = \frac{1 - ka/2}{1 + ka/2}u^n$$

and this is stable when

$$\left|\frac{1 - ka/2}{1 + ka/2}\right| < 1,$$

which holds for all $k > 0$. (This is Crank-Nicolson.)

2. (10) Consider the parabolic initial boundary value problem

$$u_t - u_{xx} = \cos(tx)$$

where $u : [0,1] \times [0,\infty) \to \mathcal{R}$, $u(0,t) = u(1,t) = 0$, and $u(x,0)$ is given. Suppose we use piecewise linear finite elements in the x variable to discretize, with nodes at $x_j = jh$, $j = 0, \ldots, M$. The jth basis function is ϕ_j, where

$$\phi_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}} & x \in [x_{j-1}, x_j] \\ \frac{x - x_{j+1}}{x_j - x_{j+1}} & x \in [x_j, x_{j+1}] \\ 0 & \text{otherwise} \end{cases}$$

Let $u_j(t) \approx u(jh,t)$ and form a system of equations for $u(t) = [u_1(t), \ldots, u_{M-1}(t)]^T$ (Don't discretize in t.) Write down the resulting system of ordinary differential equations. Give computable expressions for every matrix and vector coefficient. (These coefficients will involve integrals. For full credit, give explicit expressions for the tightest possible limits of integration. In other words, don't integrate over intervals where you know the function is zero.)

Answer: After I wrote the answer originally posted, I changed the problem. Here is the answer to the problem as given.

$$Bu'(t) + Au(t) = g(t)$$

where A and B are tridiagonal matrices of size $(M-1) \times (M-1)$.
Formulas: $f(x,t) = \cos(tx)$,

$$g_k = (f, \phi_k) = \int_{x_{k-1}}^{x_{k+1}} f(x,t)\phi_k(x)dx$$

$$b_{k\ell} = (\phi_k, \phi_\ell)$$

If $k = \ell - 1$, then this is

$$\frac{1}{h^2}\int_{x_k}^{x_{k+1}} (x - x_k)(x - x_{k+1})dx = \frac{1}{h^2}\int_0^h y(y-h)dy = \frac{1}{h^2}(h^3/3 - h^3/2) = -\frac{1}{6}h,$$

and by symmetry, the sub- and super-diagonal elements all equal $-h/6$. For the main diagonal, $b_{kk} = h/3$ for $1 \le k \le M-1$.

The matrix A is an old friend, with elements again evaluated by integration. The derivatives of ϕ_k are $\pm 1/h$, so the main diagonal elements are $2/h$, and the sub- and super diagonal elements are $-1/h$.