1. (10) Consider the equation

\[ u_t + 2u_x - 2u_y + 6u = 5(x + y + t)^2 \]

where the domain \( \Omega \) is the unit circle, and \( u : \Omega \times \mathbb{R}_+ \to \mathbb{R} \). Draw the unit circle, and mark on it the points that define the inflow boundary for this problem. Justify your answer.

**Answer:** The inflow boundary is the set of points on the unit circle satisfying \( a \cdot n < 0 \), where \( n \) is the exterior normal and \( a = [2, -2]^T \). Therefore, we need \( 2n_1 - 2n_2 < 0 \) or \( n_1 < n_2 \). This is satisfied for points on the unit circle corresponding to \( \pi/4 < \theta < 5\pi/4 \).
2. (10) Consider the problem
\[ u_{tt} - c^2 \Delta u = e^{-i\omega t} f(x) \]
with initial conditions \( u(x, 0) = u_t(x, 0) = 0 \) for \( x \in \Omega \subset \mathbb{R}^d \).

a. Assume that the solution is of the form \( u = e^{-i\omega t} z(x) \). Substitute this solution into the equation to obtain a problem of the form of the Helmholtz equation
\[ -\Delta z - \kappa^2 z = g \]
with \( z \) given on the boundary of \( \Omega \). How are \( g \) and \( \kappa \) defined?

b. Let \( \kappa = 4 \) and let \( \Omega \) be the square \((-1, 1) \times (-1, 1)\). Suppose we make a Galerkin finite element approximation to this problem. This gives us a linear system of equations to solve. In Homework 3, you solved a similar system of equations using conjugate gradients. Why can’t conjugate gradients be used on our new linear system?

**Answer:**

a. With this definition of \( u \), we see that \( u_{tt} = (-i\omega)^2 e^{-i\omega t} z(x) \), so our equation becomes
\[ (-i\omega)^2 e^{-i\omega t} z(x) - c^2 e^{-i\omega t} \Delta z(x) = e^{-i\omega t} f(x), \]
so
\[ -\Delta z - \frac{\omega^2}{c^2} z = \frac{1}{c^2} f. \]
Therefore, \( \kappa = \omega/c \) and \( g(x) = f(x)/c^2 \).

b. We know from Homework 3 that the Laplacian on the given domain has eigenvalues less than 16 and bigger than 16. Therefore, the matrix approximation to \( \Delta - \kappa^2 I \) with \( \kappa^2 = 16 \) will have both positive and negative eigenvalues. Conjugate gradients requires a positive definite matrix and therefore cannot be applied.