AMSC/CMSC 661 Scientific Computing II
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Solution of Hyperbolic Partial Differential Equations
Part 1: Theory
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These notes are based on the 2003 textbook
of Stig Larsson and Vidar Thomée.

Recall from Introduction to our Course:

Consider a differential equation that is a function of two variables, x and t:

$$au_{tt} + 2bu_{xt} + cu_{xx} + \ldots = f(x,t)$$

where the dots denote terms that have fewer than 2 derivatives. We classify the differential equation depending on a, b, and c:

- elliptic if $ac - b^2 > 0$.
  Example: Poisson's equation $u_{tt} + u_{xx} = f(x,t)$.
- hyperbolic if $ac - b^2 < 0$.
  Example: the wave equation $u_{tt} - u_{xx} = f(x,t)$.
- parabolic if $ac - b^2 = 0$.
  Example: the heat equation $u_t - u_{xx} = f(x,t)$.

It is now time to study hyperbolic equations.

The plan:

First, some of the theory (Chapter 11).
Hyperbolic equations come in three forms:

- The wave equation
- First order scalar equations
- Symmetric hyperbolic systems

What makes all of them different from

- elliptic equations (where the solution at each point is coupled to every other point)
- and parabolic equations (in which the solution now depends on what happens at every point in history)

is their dependence on only a small part of the historical data.

The IBVP for the wave equation

Reference: Section 11.2

The equation:

$$\begin{aligned} u_{tt} - \Delta u(x,t) &= 0 && \text{for } x \in \Omega \subset \mathcal{R}^d, t \in \mathcal{R}_+ \\ u(x,0) &= v(x) && \text{for } x \in \Omega \\ u_t(x,0) &= s(x) && \text{for } x \in \Omega \\ u(x,t) &= 0 && \text{for } x \in \Gamma(\Omega), t \in \mathcal{R}_+ \end{aligned}$$

This looks very much like the Initial-Boundary Value Problem for the heat equation, except for the parts in blue.

The right tool for analyzing the IBVP for the heat equation was the eigendecomposition, and we use it here, too.

Let's pull out our complete set of orthonormal eigenfunctions for $-\Delta u = \lambda u$ on Ω.

Call the eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and call the eigenfunctions z_j, so that

$$-\Delta z_j = \lambda_j z_j \text{ in } \Omega,$$

and

$$z_j = 0 \text{ on } \Gamma.$$

We'll separate variables and try to express u as a sum of z's:

$$u(x,t) = \sum_{j=1}^{\infty} w_j(t) z_j(x).$$

Let's see if we can find functions $w_j(t)$ to make this work.

Note that the boundary conditions are automatically satisfied.

Let's try to satisfy the PDE:

$$\begin{aligned} u_{tt} &= \sum_{j=1}^{\infty} w_j''(t) z_j(x) \\ -\Delta u &= \sum_{j=1}^{\infty} w_j(t) \lambda_j z_j(x) \end{aligned}$$

so

$$u_{tt} - \Delta u = \sum_{j=1}^{\infty} (w_j''(t) + \lambda_j w_j(t)) z_j(x) = 0.$$

Since the z_j form an orthonormal basis, we must have

$$w_j'' + \lambda_j w_j = 0$$

for $t > 0$, with

$$w_j(0) = (v, z_j),$$
$$w_j'(0) = (s, z_j)$$

and we know the solution to ODEs that look like this:

$$w_j(t) = (v, z_j)\cos(t\sqrt{\lambda_j}) + \frac{(s, z_j)}{\sqrt{\lambda_j}}\sin(t\sqrt{\lambda_j}),$$

so

$$u(x,t) = \sum_{j=1}^{\infty}((v, z_j)\cos(t\sqrt{\lambda_j}) + \frac{(s, z_j)}{\sqrt{\lambda_j}}\sin(t\sqrt{\lambda_j}))z_j(x).$$

Therefore, we know that a solution exists as long as this series converges.

Well-posedness

We are one step toward establishing well-posedness. We still need uniqueness and stability, consequences of this theorem:

Theorem: If u is sufficiently smooth, then the energy of the solution

$$\mathcal{E}(t) = \frac{1}{2}\int_{\Omega}(u_t(x,t)^2 + |\bigtriangledown u(x,t)|^2)dx$$

is constant with respect to time.

Proof: Theorem 11.2, p167.

Unquiz: Explain how this implies uniqueness and stability.

The cone of influence

Suppose we are interested in the solution to the wave equation at a single point $(\bar{x}, \bar{t})$, where $t > 0$.

Question: How much initial data do we need in order to compute the solution at this point?

The answer is somewhat surprising, given our experience with elliptic and parabolic equations.

Answer: We need the initial data in a d-dimensional sphere of radius $\bar{t}$ around $\bar{x}$.

Proof: See Theorem 11.3, p.167. (but not very intuitive).

First order scalar equations and characteristics

Reference: Section 11.3 is not easy to read. We'll do it by example.

This form of the problem looks like

$$u_t + a \cdot \bigtriangledown u + a_0 u = f$$

for $x \in \Omega \subset \mathcal{R}^d$, $t \in \mathcal{R}_+$, with initial conditions

$$u(x, 0) = v(x)$$

for $x \in \Omega$.

The boundary conditions are rather special:

$$u(x, t) = g(x, t)$$

for (x, t) on the inflow boundary.

- This boundary is defined by the points $x \in \Gamma(\Omega)$ and $t > 0$ for which $a \cdot n < 0$.
- n, which depends on x, is the exterior normal, the unit vector pointing out from Ω, perpendicular to Γ.

We don't specify boundary conditions on the rest of the boundary; if we do, the solution may fail to exist.

Assume that there is no point x for which $a = 0$, and assume that all of the coefficients are smooth.

The method of characteristics

Define the characteristic, or streamline, to be the solution $x(s)$ to the system of ordinary differential equations

$$\frac{dx_j(s)}{ds} = a_j(x(s)).$$

This gives us a set of coordinates $x(s)$ for every set of initial values $x(0)$.

Now back to our problem.

$$u_t + a \cdot \bigtriangledown u + a_0 u = f$$

Let $w(s) = u(x(s), s)$. Then the Chain Rule tells us that

$$\frac{dw}{ds} = u_t + \bigtriangledown u \cdot \frac{dx}{ds} = u_t + a \cdot \bigtriangledown u,$$

so our problem becomes

$$w_s + a_0 w = f$$

with $w(0) = v(x_0)$ for each x_0 on the inflow boundary.

This is just an IVP ordinary differential equation. Most amazingly, the solution along the entire characteristic depends only on the single initial value $v(x_0)$ at the point where the characteristic starts.

Examples: See p.171-3.

Jargon:

- Points at which the characteristics enter Ω are part of the inflow boundary. (Γ_- in the book)
- Points at which the characteristics leave Ω are part of the outflow boundary. (Γ_+)
- Points for which $n(x) \cdot a(x) = 0$ are on the characteristic boundary (Γ_0)

Symmetric hyperbolic systems

Reference: Section 11.4

For $x \in \mathcal{R}$ and $t \geq 0$, let $u, f : \mathcal{R}^2 \to \mathcal{R}^n$ satisfy

$$u_t + A(x,t)u_x + B(x,t)u = f(x,t)$$

with initial values $u(x,0) = v(x)$.

Suppose A, B, and f are smooth functions.

If $A = P\Lambda P^T$ is symmetric with distinct eigenvalues λ_j $(j = 1, \ldots, n)$ then we say that the system is strictly hyperbolic.

P is the matrix with the eigenvectors of A as its columns, so $P^T P = I$.

$$u_t + A(x,t)u_x + B(x,t)u = f(x,t)$$

Let's change variables: $w = P^T u$ and multiply our equation by P^T. Then

$$P^T u_t + \Lambda P^T u_x + P^T B u = P^T f.$$

Now $u = Pw$, so

$$\begin{aligned} P^T u_t &= P^T(Pw_t + P_t w) = w_t + P^T P_t w \\ P^T u_x &= P^T(Pw_x + P_x w) = w_x + P^T P_x w \end{aligned}$$

so our equation becomes

$$w_t + P^T P_t w + \Lambda(w_x + P^T P_x w) + P^T B P w = P^T f,$$

or

$$w_t + \Lambda w_x + (P^T B P + P^T P_t + \Lambda P^T P_x) w = P^T f \equiv \tilde{f}$$

which looks like the original equation except that the w_x coefficient is diagonal.

$$w_t + \Lambda w_x + \tilde{B} w = \tilde{f}$$

Case 1: $\tilde{B} = 0$. Then we have n uncoupled ODEs

$$(w_j)_t + \lambda_j (w_j)_x = \tilde{f}_j,$$

with $w_j(x, 0)$ given.

We know how to solve these equations using the method of characteristics

Once we have the solution, we form $u = Pw$ and we are done.

Therefore, we have existence of the solution. Well-posedness also holds.

Example: p177.

Case 2: $\tilde{B} \neq 0$. Then we can't explicitly write the solution, but your book shows that the problem is well posed.

Conclusion

We have defined well-posed problems in three forms:

- The wave equation, analyzed by eigenfunctions of $-\Delta u$.
- First order scalar equations, analyzed by characteristics.
- Symmetric hyperbolic systems, decoupled by matrix eigenvectors and solved by characteristics.

Next: Numerical methods.