The plan:

Recall: Hyperbolic equations come in three forms:

- First order scalar equations, analyzed by characteristics. We’ll use finite differences.
- Symmetric hyperbolic systems, decoupled by matrix eigenvectors and solved by characteristics. We’ll use finite differences.
- The wave equation, analyzed by eigenfunctions of $-\Delta u$. We’ll use finite elements in class and finite differences in the homework.

Stability proofs are similar to those for parabolic equations, and we will omit them.

Special challenges from hyperbolic equations

- Discontinuities are preserved, so they must be approximated well.
- Conservation of energy is important and (ideally) should be preserved by the numerical method.

First order scalar equations and characteristics

What we know

$$u_t + a \cdot \nabla u + a_0 u = f$$

for $x \in \Omega \subset \mathbb{R}^d$, $t \in \mathbb{R}_+$, with initial conditions

$$u(x, 0) = v(x)$$

for $x \in \Omega$, and boundary conditions

$$u(x, t) = g(x, t)$$

for $(x, t)$ on the inflow boundary.

Assume that there is no point $x$ for which $a = 0$, and assume that all of the coefficients are smooth.
Define the characteristic, or streamline, to be the solution $x(s)$ to the system of ordinary differential equations

$$\frac{dx_j(s)}{ds} = a_j(x(s)).$$

The solution along the entire characteristic depends only on the single initial value $v(x_0)$ at the point where the characteristic starts.

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**Numerical methods**

The simplest possible problem is

$$u_t = au_x,$$

with

$$u(x,0) = v(x).$$

Note that we have switched the sign of $a$ from the previous slide. We could use finite differences on both terms:

$$\frac{u_j^{n+1} - u_j^n}{k} \frac{u_{j+1}^n - u_j^n}{h} = a,$$

yielding

$$u_j^{n+1} = aru_{j+1}^n + (1-ar)u_j^n$$

$(r = k/h)$, which (we see) is stable if $1 - ar > 0$.

This finite difference method is 1st order in $k$ and $h$.

**Unquiz:** Try this method

$$u_j^{n+1} = aru_{j+1}^n + (1-ar)u_j^n$$

with $a = 1$ and $v(x)$ a step function that is 1 for $1/2 \leq x \leq 1$ and 0 elsewhere.

Then try it for $a = -1$ and the same $v(x)$.

How well does it work? [ ]

From the 2nd part of the unquiz, we see that it is important to take the $x$ difference in the upwind direction.

The general principle is called the Courant-Friedrichs-Lewy (CFL) condition for stability: in order for a method to be stable, it is necessary that the domain of dependence for the numerical method contain the domain of dependence for the differential equation.

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**Higher order methods**
Note that we can construct methods of $O(h^p)$ by taking more than 2 points in our approximation to $u_x$. See pp. 187-189

It would be natural to use a symmetric difference instead of an upwind one. Unfortunately, this method is unstable for all choices of $r$ unless we also use a more complicated approximation to $u_t$.

Some useful difference methods

Friedrichs method

$$\frac{u_j^{n+1} - u_j^n}{k} = a \frac{u_{j+1}^n - u_j^n}{2h} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2k}$$

Notice that the last term is $h/(2r)$ times an approximation to $u_{xx}$. This addition of an artificial diffusion term (sometimes called artificial viscosity) is a common way to stabilize methods. The Friedrichs method is first order accurate and requires $-1 \leq ar \leq 1$ for stability.

Lax-Wendroff method

$$u_j^{n+1} = \alpha u_{j+1}^n + \beta u_j^n + \gamma u_{j-1}^n$$

with

$$\alpha = \frac{a^2r^2 + ar}{2},$$

$$\beta = 1 - a^2r^2,$$

$$\gamma = \frac{a^2r^2 - ar}{2}.$$ 

This is stable in the $L_2$ norm when $a^2r^2 \leq 1$ but not stable in the max-norm. It is 2nd order accurate in space.

Wendroff box method for $u_t + au_x + a_0u = f$

$$\frac{u_j^{n+1} + u_{j+1}^{n+1} - u_j^n - u_{j+1}^n}{2k} + a \frac{u_{j+1}^{n+1} + u_{j+1}^n - u_j^{n+1} - u_j^n}{2h} + a_0 \frac{u_j^{n+1} + u_{j+1}^{n+1} + u_j^n + u_{j+1}^n}{4} = f_j^n$$

This is stable in $L_2$ when $1 - ar > 0$, and 2nd order accurate in space.

Symmetric hyperbolic systems

For $x \in \mathcal{R}$ and $t \geq 0$, let $u, f : \mathcal{R}^2 \rightarrow \mathcal{R}^n$ satisfy

$$u_t + A(x, t)u_x + B(x, t)u = f(x, t)$$
with initial values \( u(x, 0) = v(x) \).

Suppose \( A, B, \) and \( f \) are smooth functions.

The methods we discussed for the scalar equation all have generalizations to symmetric hyperbolic systems. See Section 12.2.

The IBVP for the wave equation


The equation:

\[
\begin{align*}
\frac{∂^2 u}{∂t^2} - \Delta u(x, t) &= 0 & \text{for } x \in Ω \subset \mathbb{R}^2, t \in \mathbb{R}_+ \\
u(x, 0) &= v(x) & \text{for } x \in Ω \\
u_t(x, 0) &= s(x) & \text{for } x \in Ω \\
u(x, t) &= 0 & \text{for } x \in Γ(Ω), t \in \mathbb{R}_+
\end{align*}
\]

Assume that the boundary of \( Ω \subset \mathbb{R}^2 \) is a convex polygon.

Let’s use piecewise linear finite elements, expressing

\[
u_h(x, t) = \sum_{i=1}^{M} α_j(t)φ_j(x)
\]

where \( φ_j \) is a hat function.

In weak form, our equation is

\[
((u_h)_{tt}, φ) + a(u_h, φ) = (f, φ),
\]

\[
u_h(x, 0) = v_h(x), \quad (u_h)_t(x, 0) = s_h(x),
\]

where \( v_h \) and \( s_h \) are piecewise linear approximations to \( v \) and \( s \):

\[
v_h = \sum_{j=1}^{M} β_jφ_j(x), \quad s_h = \sum_{j=1}^{M} γ_jφ_j(x),
\]

This gives us a system of ordinary differential equations for the coefficients \( α \):

\[
Bα''(t) + Aα(t) = f(t)
\]

where

\[
b_{mj} = (φ_m, φ_j) \]
\[
a_{mj} = a(φ_m, φ_j) \]
\[
f_m = (f, φ_m)
\]
and the initial conditions are
\[ \alpha(0) = \beta, \quad \alpha'(0) = \gamma. \]

**Energy conservation**

Recall that for the wave equation, the energy
\[ E(t) = \frac{1}{2} \int_{\Omega} u_t^2 + |\nabla u|^2 \, dx \]
is constant when \( f = 0 \). Is this true for our discrete problem?

Let’s take our weak formulation
\[ ((u_h)_{tt}, \phi) + a(u_h, \phi) = 0 \]
This is true for all piecewise linear functions, so let’s use \((u_h)_t\):
\[ ((u_h)_{tt}, (u_h)_t) + a(u_h, (u_h)_t) = 0. \]

Now notice that this is just
\[ \frac{1}{2} \frac{d}{dt} (\| (u_h)_t \|^2 + |u_h|^2) = 0, \]
so the energy is conserved.

**Error formula**

**Theorem:** (13.2, p204) Given a function \( g(x) \), Let \( R_h g \) be the piecewise linear function that interpolates at the meshpoints. If we take the initial values
\[ u^0 = R_h v, \]
\[ u^1 = R_h (v + ks + \frac{k^2}{2} (\Delta v + f(\cdot, 0))), \]
then
\[ \| u^{n+1/2} - u(t_n + k/2) \| + \| (u^{n+1} - u^n)/k - u_{t}(t_n + k/2) \| \leq C(h^2 + k^2), \]
where \( C \) depends on \( u \) and \( t_n \).

Also,
\[ |u^{n+1/2} - u(t_n + k/2)|_1 \leq C(h + k^2). \]

**Final comments**

Solving hyperbolic equations is more difficult (in fact, more of an art) than solving parabolic or elliptic:

- Many obvious difference formulas are unstable.
• You need to know a fair amount about the solution before you begin; for example, characteristic directions, inflow boundaries, domain of dependence.

• The solution does not smooth out with time, so shock waves and other discontinuities persist.

• Conservation of energy is an important consideration.

We have just touched the surface of this subject.